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ORIENTED SUPERSINGULAR ELLIPTIC CURVES & CLASS GROUP ACTIONS

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ISOGENY GRAPHS

Definition

Given an elliptic curve E over k, and a finite set of primes S, we can associate an isogeny graph $G_S(E)$

- whose vertices are elliptic curves isogenous to E over $\bar{k}$, and
- whose edges are isogenies of degree $\ell \in S$.

The vertices are defined up to $\bar{k}$-isomorphism and the edges from a given vertex are defined up to a $\bar{k}$-isomorphism of the codomain.

If $S = \{\ell\}$, then we call G an ℓ-isogeny graph, G_ℓ.

For an elliptic curve E/k and prime $\ell \neq \texttt{char}(k)$, the full ℓ-torsion subgroup is a 2-dimensional $\mathbb{F}_\ell$-vector space:

$$E[\ell] = \left\{ P \in E[\bar{k}] \,\middle|\, \ell P = O \right\} \simeq \mathbb{F}_\ell^2$$

Consequently, the set of cyclic subgroups is in bijection with $\mathbb{P}^1(\mathbb{F}_\ell)$, which in turn are in bijection with the set of ℓ-isogenies from E.

Thus, the ℓ-isogeny graph of E is $(\ell + 1)$-regular (as a directed multigraph).

SUPERSINGULAR ISOGENY GRAPHS

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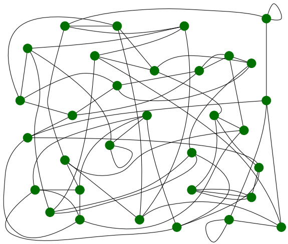
The supersingular isogeny graphs are remarkable because the vertex sets are finite : there are $(p+1)/12 + \epsilon_p$ curves. Moreover

- every supersingular elliptic curve can be defined over $\mathbb{F}_{p^2}$;
- all ℓ-isogenies are defined over $\mathbb{F}_{p^2}$;
- every endomorphism of E is defined over $\mathbb{F}_{p^2}$.

The lack of a commutative group acting on the set of supersingular elliptic curves/$\mathbb{F}_{p^2}$ makes the isogeny graph more complicated.



SUPERSINGULAR ISOGENY GRAPHS - SPECIAL VERTICES

Supersingular curves with j-invariants 0 and 1728 have extra automorphisms, besides $[\pm 1]$.

- E_{1728} is supersingular if $p \equiv 3 \mod 4$

$$\text{Aut}(E_{1728}) = \{[\pm 1], [\pm i]\} \qquad \text{End}(E_{1728}) = \mathbb{Z}\langle [i], \frac{1+\pi_p}{2} \rangle$$

  where $[i](x, y) = (-x, iy)$ for $i^2 = -1$ in $\mathbb{F}_{p^2}$ and $\pi_p(x, y) = (x^p, y^p)$ is Frobenius.

- E_0 is supersingular if $p \equiv 2 \mod 3$

$$\text{Aut}(E_0) = \{[\pm 1], [\pm \zeta_3], [\pm \zeta_3^2]\} \qquad \text{End}(E_0) = \mathbb{Z}\langle [\zeta_3], \pi_p \rangle$$

  where $[\zeta_3](x, y) = (\zeta_3 x, y)$ for $\zeta_3^2 + \zeta_3 + 1 = 0$ in $\mathbb{F}_{p^2}$.

Because of these extra automorphisms, supersingular isogeny graphs may fail to really be undirected graphs.
Since this issue occurs only at neighbours of E_0 and E_{1728}, we usually forget this subtlety.

ORIENTATIONS

Let $\mathcal{O}$ be an order in an imaginary quadratic field K. An *$\mathcal{O}$-orientation* on a supersingular elliptic curve E is an embedding $\iota : \mathcal{O} \hookrightarrow \mathsf{End}(E)$, and a *$K$-orientation* is an embedding $\iota : K \hookrightarrow \mathsf{End}^0(E) = \mathsf{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. An $\mathcal{O}$-orientation is *primitive* if $\mathcal{O} \simeq \mathsf{End}(E) \cap \iota(K)$.

Theorem

The category of K-oriented supersingular elliptic curves (E, ι), whose morphisms are isogenies commuting with the K-orientations, is equivalent to the category of elliptic curves with CM by K.

Let $\phi : E \to F$ be an isogeny of degree ℓ. A K-orientation $\iota : K \hookrightarrow \mathsf{End}^0(E)$ determines a K-orientation $\phi_*(\iota) : K \hookrightarrow \mathsf{End}^0(F)$ on F, defined by

$$\phi_*(\iota)(\alpha) = \frac{1}{\ell}\, \phi \circ \iota(\alpha) \circ \hat{\phi}.$$

Conversely, given K-oriented elliptic curves (E, ι_E) and (F, ι_F) we say that an isogeny $\phi : E \to F$ is K-oriented if $\phi_*(\iota_E) = \iota_F$, i.e., if the orientation on F is induced by ϕ.

ORIENTED ISOGENY GRAPHS - VERTICES & EDGES

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Two K-oriented curves are isomorphic if and ony if there exists a K-oriented isomorphism between them. We denote $G_S(E,K)$ the S-isogeny graph of K-oriented supersingular elliptic curves over $\mathbb{F}_{p^2}$ whose vertices are isomorphism classes of K-oriented supersingular elliptic curves $/\mathbb{F}_{p^2}$ and whose edges are equivalence classes of K-oriented isogenies of degree in S.

Proposition*

The only vertices of $G_\ell(E,K)$ with extra automorphisms are (E,ι) where either

- $E = E_{1728}$ and $\iota(i) = [\pm i]$ or
- $E = E_0$ and $\iota(\zeta_3) = [\pm\zeta_3]$.

Then (E,ι) has out-degree $\ell+1$, except at the oriented curves with extra automorphisms, in which case thise degree is $2(\ell+1-r_\ell)/|\mathrm{Aut}(E)|+r_\ell$ where $|\mathrm{Aut}(E)|r_\ell$ is the number of elements of $\mathcal{O}$ of norm ℓ.

*Arpin, S. and Chen, M. and Lauter, K.E. and Scheidler, R. and Stange, K.E. and Tran, H.T.N. - Orientations and cycles in supersingular isogeny graphs

Oriented isogeny graphs - structure

The orientation by a quadratic imaginary field gives to supersingular isogeny graphs the rigid structure of a volcano. It also differentiates vertices in the descending paths from the crater, determining an infinite graph.

$G_\ell(E, K)$ consists of connected components, each of which is a volcano.

- The crater consists of K-oriented elliptic curves which are $\mathcal{O}$-primitive for some fixed ℓ-fundamental order $\mathcal{O}$ of K.
- Oriented curves at depth k are primitively oriented by orders of index ℓ^k in $\mathcal{O}$.
- We recover the standar terminology for oriented isogenies:
  - If $\mathcal{O} = \mathcal{O}'$ we say that ϕ is horizontal;
  - If $\mathcal{O} \supsetneq \mathcal{O}'$ we say that ϕ is ascending;
  - If $\mathcal{O} \subsetneq \mathcal{O}'$ we say that ϕ is descending.

ORIENTED ISOGENY GRAPHS - AN EXAMPLE

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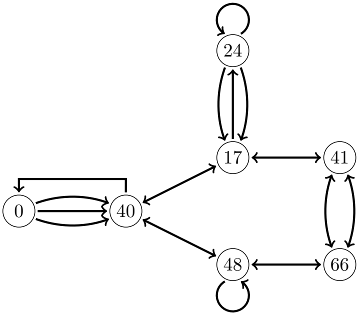
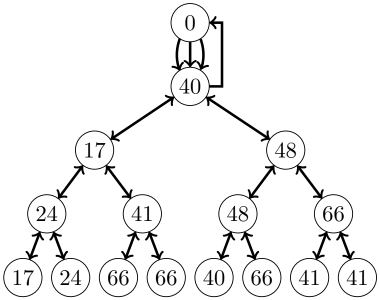
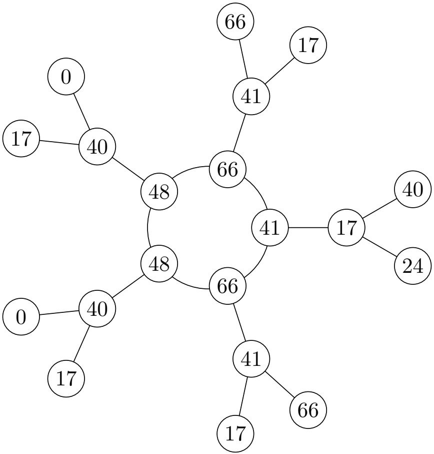
Let $E_0/\mathbb{F}_{71}$ be the supersingular elliptic curve with $j(E) = 0$, oriented by the order $\mathcal{O}_K = \mathbb{Z}[\omega]$, where $\omega^2 + \omega + 1 = 0$. The unoriented 2-isogeny graph is the finite graph on the left.
The orientation by $K = \mathbb{Q}[\omega]$ differentiates vertices in the descending paths from E_0, determining an infinite graph shown here to depth 4:





ORIENTED ISOGENY GRAPHS - YET ANOTHER EXAMPLE

We let again $p = 71$ and we consider the isogeny graph oriented by $\mathbb{Z}[\omega_{79}]$ where ω_{79} generates the ring of integers of $\mathbb{Q}(\sqrt{-79})$.



ISOGENY CHAINS

Definition

An ℓ-isogeny chain of length n from E_0 to E is a sequence of isogenies of degree ℓ:

$$E_0 \xrightarrow{\phi_0} E_1 \xrightarrow{\phi_1} E_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{n-1}} E_n = E.$$

The ℓ-isogeny chain is without backtracking if $\texttt{ker}\,(\phi_{i+1} \circ \phi_i) \neq E_i[\ell],\ \forall i$. The isogeny chain is descending (or ascending, or horizontal) if each ϕ_i is descending (or ascending, or horizontal, respectively).

The dual isogeny of ϕ_i is the only isogeny ϕ_{i+1} satisfying $\texttt{ker}\,(\phi_{i+1} \circ \phi_i) = E_i[\ell]$. Thus, an isogeny chain is without backtracking if and only if the composition of two consecutive isogenies is cyclic.

Lemma

The composition of the isogenies in an ℓ-isogeny chain is cyclic if and only if the ℓ-isogeny chain is without backtracking.

PRIMITIVE ORIENTATIONS

- $\mathsf{SS}(p) = \{$supersingular elliptic curves over $\overline{\mathbb{F}}_p$ up to isomorphism$\}$.
- $\mathsf{SS}_{\mathcal{O}}(p) = \{\mathcal{O}$-oriented s.s. elliptic curves over $\overline{\mathbb{F}}_p$ up to K-isomorphism$\}$.
- $\mathsf{SS}_{\mathcal{O}}^{pr}(p) =$ subset of primitive $\mathcal{O}$-oriented curves.

An element of $\mathsf{SS}_{\mathcal{O}}^{pr}(p)$ consists of

- A supersingular elliptic curve $E/\overline{\mathbb{F}}_p$;
- a primitive orientation $\iota : \mathcal{O} \hookrightarrow \mathsf{End}(E)$;
- a structure of a p-orientation which is a homomorphism $\rho : \mathcal{O} \to \overline{\mathbb{F}}_p$.

Note that $\mathsf{End}(E)$ is equipped with a p-orientation $\rho : \mathsf{End}(E) \hookrightarrow \overline{\mathbb{F}}_p$.

We denote by $\mathsf{SS}_{\mathcal{O}}(\rho)$ the set of oriented supersingular elliptic curves with ρ induced by ι and $\mathsf{End}(E)/\mathfrak{P} \hookrightarrow \overline{\mathbb{F}}_p$, and $\mathsf{SS}_{\mathcal{O}}(\overline{\rho})$ the opposite p-orientation class.

THE SET OF PRIMITIVE p-ORIENTED CURVES

Since $K \hookrightarrow \mathfrak{A}$, then

- If p splits in K then $\mathsf{SS}_{\mathcal{O}}^{pr}(p)$ is empty.
- If p ramifies in K then $\rho = \overline{\rho}$ and then

$$\mathsf{SS}_{\mathcal{O}}^{pr}(p) = \mathsf{SS}_{\mathcal{O}}^{pr}(\rho) = \mathsf{SS}_{\mathcal{O}}^{pr}(\overline{\rho})$$

- If p is inert in K then $\mathsf{SS}_{\mathcal{O}}(p)$ decomposes in

$$\begin{aligned}\mathsf{SS}_{\mathcal{O}}^{pr}(p) &= \mathsf{SS}_{\mathcal{O}}^{pr}(\rho) \cup \mathsf{SS}_{\mathcal{O}}^{pr}(\overline{\rho}) \\ &= \mathsf{SS}_{\mathcal{O}}^{pr}(\rho) \cup \mathsf{SS}_{\mathcal{O}}^{pr}(\rho)^{\sigma}\end{aligned}$$

CLASS GROUP ACTION

The set $\mathsf{SS}_{\mathcal{O}}(p)$ admits a transitive group action:

$$\begin{aligned} \mathcal{C}\ell(\mathcal{O}) \times \mathsf{SS}_{\mathcal{O}}(p) &\longrightarrow \mathsf{SS}_{\mathcal{O}}(p) \\ ([\mathfrak{a}], E) &\longmapsto [\mathfrak{a}] \cdot E = E/E[\mathfrak{a}] \end{aligned}$$

Proposition

The set $\mathsf{SS}_{\mathcal{O}}^{pr}(\rho)$ is a torsor for the class group $\mathcal{C}\ell(\mathcal{O})$.

In particular, for fixed primitive p-oriented supersingular elliptic curve E, we obtain a bijection of sets:

$$\begin{aligned} \mathcal{C}\ell(\mathcal{O}) &\longrightarrow \mathsf{SS}_{\mathcal{O}}^{pr}(\rho) \\ [\mathfrak{a}] &\longmapsto [\mathfrak{a}] \cdot E \end{aligned}$$

EFFECTIVE CLASS GROUP ACTION

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The theory of complex multiplication relates the geometry of isogenies to the arithmetic Galois action on elliptic curves in characteristic zero, mediated by the map of $\mathcal{C}\ell(\mathcal{O})$ into each.

Over a finite field, we use the geometric action by isogenies to recover the class group action. In particular we describe the action of $\mathcal{C}\ell(\mathcal{O})$ on ℓ-isogeny chains in the *whirlpool*.

Suppose that (E_i, ϕ_i) is a descending ℓ-isogeny chain with

$$\mathcal{O}_K \subseteq \mathsf{End}(E_0), \dots, \mathcal{O} = \mathbb{Z} + \ell^n \mathcal{O}_K \subseteq \mathsf{End}(E_n).$$

If $\mathfrak{q}$ is a split prime in $\mathcal{O}_K$ over $q \neq \ell, p$, then the isogeny

$$\psi_0 : E_0 \to F_0 = E_0 / E_0[\mathfrak{q}]$$

can be extended to the ℓ-isogeny chain by pushing forward the cyclic group $C_0 = E_0[\mathfrak{q}]$:

$$C_0 = E_0[\mathfrak{q}], C_1 = \phi_0(C_0), \dots, C_n = \phi_{n-1}(C_{n-1}),$$

and defining $F_i = E_i / C_i$.

LADDERS

This construction motivates the following definition.

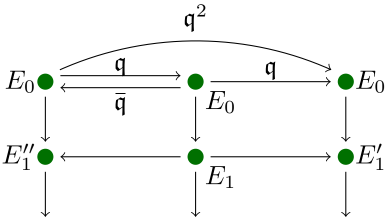
Definition

An ℓ-ladder of length n and degree q is a commutative diagram of ℓ-isogeny chains (E_i, ϕ_i), (F_i, ϕ'_i) of length n connected by q-isogenies $\psi_i : E_i \to F_i$

$$\begin{array}{ccccccccc}
E_0 & \xrightarrow{\phi_0} & E_1 & \xrightarrow{\phi_1} & E_2 & \xrightarrow{\phi_2} & \cdots & \xrightarrow{\phi_{n-1}} & E_n \\
\downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & & & \downarrow{\scriptstyle \psi_n} \\
F_0 & \xrightarrow{\phi'_0} & F_1 & \xrightarrow{\phi'_1} & F_2 & \xrightarrow{\phi'_2} & \cdots & \xrightarrow{\phi'_{n-1}} & F_n
\end{array}$$

If ψ_0 is as above ($(\psi_0) = E_0[\mathfrak{q}]$), the ladder encodes the action of $\mathcal{C}\ell(\mathcal{O})$ on ℓ-isogeny chains, and consequently on elliptic curves at depth n.

DIFFERENTIATING CONJUGATE IDEAL CLASSES



$E_i' \neq E_i''$ if and only if $\mathfrak{q}^2 \cap \mathcal{O}_i$ is not principal and the probability that a random ideal in $\mathcal{O}_i$ is principal is $1/h(\mathcal{O}_i)$. In fact, we can do better; we write $\mathcal{O}_K = \mathbb{Z}[\omega]$ and we observe that if $\mathfrak{q}^2$ was principal, then

$$q^2 = \mathsf{N}(\mathfrak{q}^2) = \mathsf{N}(a + b\ell^i\omega)$$

since it would be generated by an element of $\mathcal{O}_i = \mathbb{Z} + \ell^i\mathcal{O}_K$. Now

$$\mathsf{N}(a + b\ell^i) = a^2 \pm abt\ell^i + b^2 s\ell^{2i} \quad \text{where} \quad \omega^2 + t\omega + s = 0$$

Thus, as soon as $\ell^{2i} \gg q^2$, we are guaranteed that $\mathfrak{q}^2$ is not principal.

AN EQUIVALENCE OF CATEGORIES

Quadratic forms (a, b, c)	Ideal classes $\left[(a, \frac{-b+\sqrt{\Delta}}{2})\right]$	K-lattices $a\mathbb{Z} + \frac{-b\sqrt{\Delta}}{2}\mathbb{Z}$
Ideal classes $[\mathfrak{a}] = [(\alpha, \beta)]$	Quadratic forms $\frac{N(\alpha x - \beta y)}{N(\mathfrak{a})}$	K-lattices $\alpha\mathbb{Z} + \beta\mathbb{Z}$
K-lattices $\omega_1\mathbb{Z} + \omega_2\mathbb{Z}$	Quadratic forms $\frac{N(\omega_1 x - \omega_2 y)}{N(\omega_1)}$	Ideal classes (ω_1, ω_2)

INITIALIZING THE LADDER - AN EXAMPLE

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Suppose $D_K = -3$, and $\ell = 2$; we note that for all $n \geq 3$, that

$$\mathcal{C}\ell(\mathcal{O}_n) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2^{n-2}\mathbb{Z}$$

and in particular, $\mathcal{C}\ell(\mathcal{O}_n)[2]$ consist of the classes of binary quadratic forms

$$\{\langle 1, 0, |D_K|\ell^{2(n-1)}\rangle, \langle |D_K|, 0, \ell^{2(n-1)}\rangle, \langle \ell^2, \ell^2, n_1\rangle, \langle \ell^2|D_K|, \ell^2|D_K|, n_2\rangle\}.$$

where $\ell^4 - 4\ell^2 n_1 = \ell^4|D_K|^2 - 4\ell^2|D_K|n_2 = -\ell^{2n}|D_K|$, whence

$$n_1 = 1 + \ell^{2(n-2)}|D_K| \text{ and } n_2 = |D_K| + \ell^{2(n-2)}.$$

For $n = 3$, the form $\langle 12, 12, 7\rangle$ reduces to $\langle 7, 2, 7\rangle$ and the reduced representatives are:

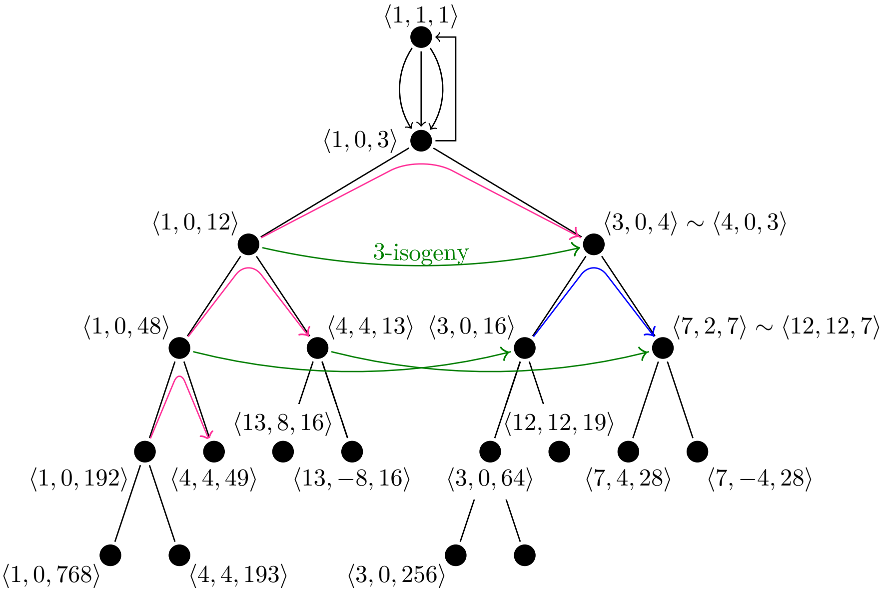
$$\{\langle 1, 0, 48\rangle, \langle 3, 0, 16\rangle, \langle 4, 4, 13\rangle, \langle 7, 2, 7\rangle\}.$$

but for for $n \geq 4$, since $12 < n_2$, the forms

$$\{\langle 1, 0, 3 \cdot 4^{n-1}\rangle, \langle 3, 0, 4^{n-1}\rangle, \langle 4, 4, n_1\rangle, \langle 12, 12, n_2\rangle\}$$

are reduced.

INITIALIZING THE LADDER - A PICTURE



INITIALIZING THE LADDER - A TABLE

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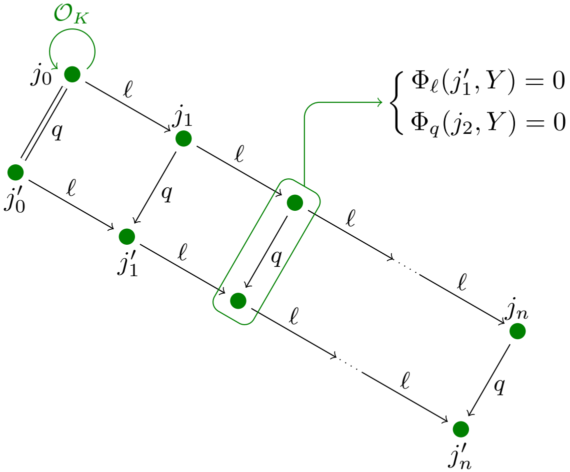
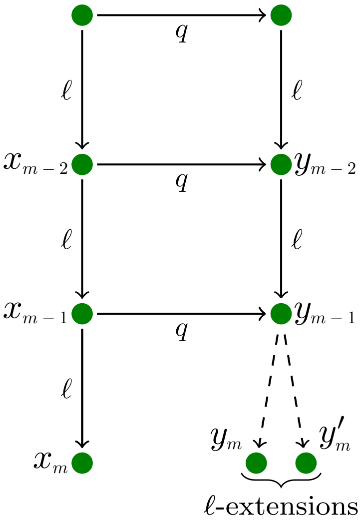
q	m	f_m	$[f_m]$	$[f_{m-1}]$
7	4	$\langle 7,4,28\rangle$	$[\langle 7,4,28\rangle]$	$[\langle 7,2,7\rangle]$
13	4	$\langle 13,8,16\rangle$	$[\langle 13,8,16\rangle]$	$[\langle 4,4,13\rangle]$
19	5	$\langle 19,14,43\rangle$	$[\langle 19,14,43\rangle]$	$[\langle 12,12,19\rangle]$
31	4	$\langle 31,10,7\rangle$	$[\langle 7,4,28\rangle]$	$[\langle 7,2,7\rangle]$
37	4	$\langle 37,34,13\rangle$	$[\langle 13,-8,16\rangle]$	$[\langle 4,4,13\rangle]$
43	5	$\langle 43,14,19\rangle$	$[\langle 19,-14,43\rangle]$	$[\langle 12,12,19\rangle]$
61	4	$\langle 61,56,16\rangle$	$[\langle 13,-8,16\rangle]$	$[\langle 4,4,13\rangle]$
67	6	$\langle 67,24,48\rangle$	$[\langle 48,-24,67\rangle]$	$[\langle 12,12,67\rangle]$
73	5	$\langle 73,40,16\rangle$	$[\langle 16,-8,49\rangle]$	$[\langle 4,4,49\rangle]$
79	4	$\langle 79,38,7\rangle$	$[\langle 7,4,28\rangle]$	$[\langle 7,2,7\rangle]$
97	5	$\langle 97,56,16\rangle$	$[\langle 16,8,49\rangle]$	$[\langle 4,4,49\rangle]$
103	4	$\langle 103,46,7\rangle$	$[\langle 7,-4,28\rangle]$	$[\langle 7,2,7\rangle]$
109	4	$\langle 109,70,13\rangle$	$[\langle 13,8,16\rangle]$	$[\langle 4,4,13\rangle]$
127	4	$\langle 127,116,28\rangle$	$[\langle 7,4,28\rangle]$	$[\langle 7,2,7\rangle]$

COMPLETING SQUARES OF ISOGENIES



EXTENDING THE LADDER



Let $\ell = 2$.

- The two ℓ-extensions are determined by a quadratic polynomial (deduced from y_{m-1}, y_{m-2}:

$$\phi_\ell(y) = 0$$

  We can solve for y_m, y'_m, its roots.

- We have a degree $q+1$ polynomial $\phi_q(y) = 0$ determined by x_m but we do note need to compute it. It suffices

$$\phi_q(y) \mod \phi_\ell(y)$$

  Indeed

$$\Phi_q(x, y) \equiv \phi_q(y) \mod (x - x_m, \phi_\ell(y))$$

ADDING LEVEL STRUCTURE

There are multiple reasons to add level structure to our construction:

- With an ℓ-level structure, the extension of ℓ-isogenies by modular correspondences allows one to automatically remove the dual isogeny (backtracking): there are ℓ rather than $\ell + 1$ extensions.

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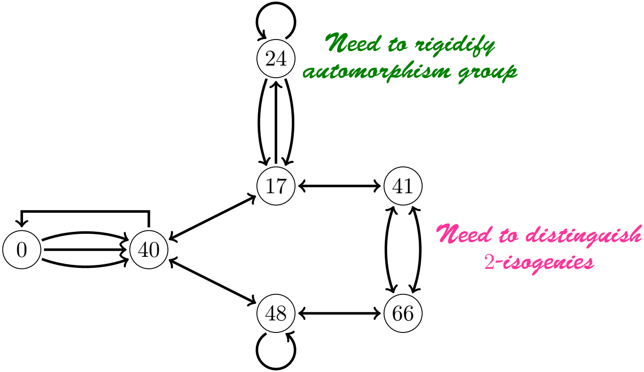
- The modular isogeny chain is a potentially-non injective image of the isogeny chain.



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- Rigidifying automorphisms should also shorten the distance to which we need to go in order to differentiate 2 points (two torsion of $\mathcal{C}\ell(\mathcal{O})$ may lift to non 2-torsion point in $\mathcal{C}\ell(\mathcal{O}, \Gamma)$).

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- q-modular polynomial of higher level are smaller.

ISOGENY GRAPHS WITH LEVEL STRUCTURE

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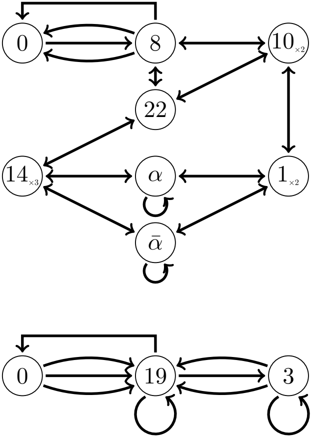
For any congruence subgroup Γ of level coprime to the characteristic, we have a covering $G_S(E,\Gamma) \to G_S(E)$ whose vertices are pairs $(E,\Gamma(P,Q))$ of supersingular elliptic curves/$\mathbb{F}_{p^2}$ and a Γ-level structure, and edges are isogenies $\psi : (E,\Gamma(P,Q)) \to (E',\Gamma(P',Q'))$ such that $\psi(\Gamma(P,Q)) = \Gamma(P',Q')$.



Eg. $\Gamma_0(N)$-structures.
Vertices (E,G) with $G \leq E[N]$ of order N
End$(E,G) = \{\alpha \in \text{End}(E) \mid \alpha(G) \subseteq G\}$ isomorphic to Eichler order.

On the left the $\Gamma_0(3)$ supersingular 2-isogeny graph.

$14 \leftrightarrow \{(E_0,G_1),(E_0,G_2),(E_0,G_3)\}$ where G_1,G_2,G_3 maps to each other under the automorphism of E_0; they define 3 isogenies to E_3.

ORIENTED ISOGENY GRAPHS WITH LEVEL STRUCTURE

We will write $G_S(\mathsf{SS}_K(p,\Gamma))$ or $G_S(\mathsf{SS}_{\mathcal{O}}(p,\Gamma))$ for the supersingular isogeny graphs oriented by K with Γ-level structure.

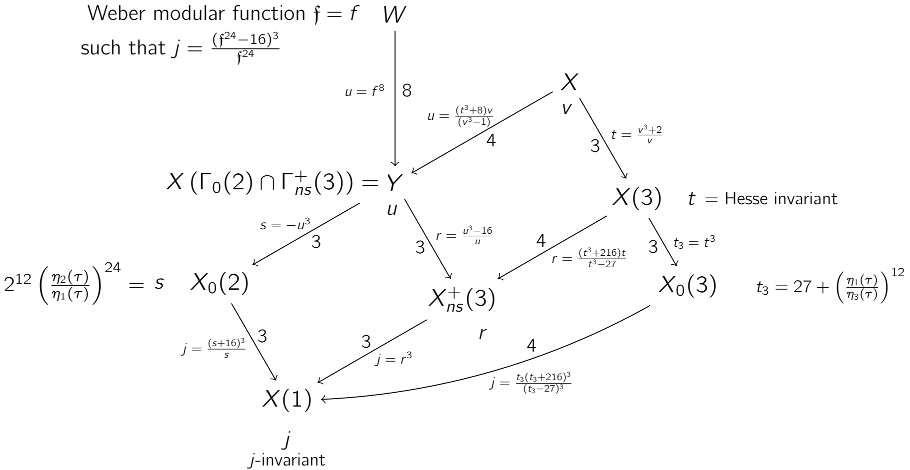
Once again we have covers

$$G_S(\mathsf{SS}_K(p,\Gamma)) \to G_S(E,K) \qquad G_S(\mathsf{SS}_{\mathcal{O}}(p,\Gamma)) \to G_S(E,\mathcal{O})$$

The action of ideals through isogenies lets us define an action on $G_S(\mathsf{SS}_{\mathcal{O}}(p,\Gamma))$ by a ray class group $\mathcal{C}\ell(\mathcal{O},\Gamma)$

$$\begin{aligned} \mathcal{C}\ell(\mathcal{O},\Gamma) \times \mathsf{SS}_{\mathcal{O}}(p,\Gamma) &\longrightarrow \mathsf{SS}(p,\Gamma) \\ ([\mathfrak{a}],(E,\Gamma(P,Q))) &\longrightarrow (\phi_{\mathfrak{a}}(E),\Gamma(\phi_{\mathfrak{a}}(P),\phi_{\mathfrak{a}}(Q))) \end{aligned}$$

SOME MODULAR CURVES OF INTEREST



WEBER INITIALIZATIONS

Let u be a supersingular value of the Weber function,

$$r = u^3 \qquad t = -u^8 \qquad s = t^3$$

along the chain $\mathcal{W}_8 \to Y \to X_0(2) \to X(1)$. we get

$$\Psi_2(x,y) = (x^2 - y)y + 16x \qquad \Psi_3(x,y) = x^4 - x^3y^3 + 8xy + y^4$$

The elliptic curves associated to Weber invariants are the fiber in the family:

$$y^2 + xy = x^3 - \frac{1}{u^{24} - 64}x$$

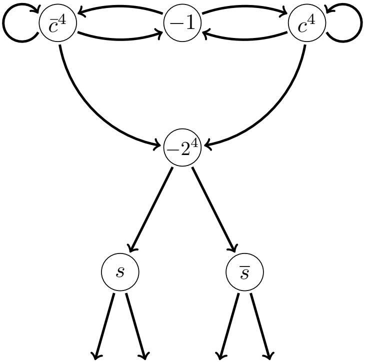
over u on the Weber curve.

The initial values with which to build the public ℓ-isogeny chains are

D	j_0	s_0	t_0	D	j_1	s_1	t_1
-3	0	-2^4	$-(\sqrt[3]{2})^4$	-12	$2^4 15^3$	-2^8	$-(\sqrt[3]{2})^8$
-4	12^3	2^3	2	-16	66^3	2^9	2^3
-7	-15^3	-1	-1	-28	255^3	-2^{12}	-2^4
-8	20^3	2^6	2^2	-32	j_1	t_1^3	$2^3(\sqrt{2}+1)$

WEBER INITIALIZATIONS - DISCRIMINANT -7

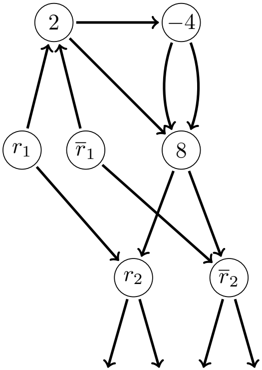
Endomorphism ring is small: generated by an endomorphism of degree 2 we avoid any pathologies associated with the extra automorphisms.



- $t_0 = -1$ and c root of $x^2 - x + 2$.
- c^4 and $\bar{c}^4$ also t-values over $j = -15^3$.
- $\Psi_2(-1, c^4) = \Psi_2(-1, \bar{c}^4) = 0$, the two extensions correspond to the horizontal 2-isogenies.
- $\Psi_2(c^4, c^4) = \Psi_2(c^4, -2^4) = 0$: the former enters a cycle the latter induces a descending isogeny.

Initialization: $(t_0, t_1, t_2, \dots)$ beginning with $(-1, c^4, -2^4, \dots)$.
Successive solutions to $\Psi_2(t_i, t_{i+1}) = 0$ are necessarily descending.
Extension: random choice of root t_{i+1} of $\Psi_2(t_i, x)$.

WEBER INITIALIZATIONS - DISCRIMINANT -4

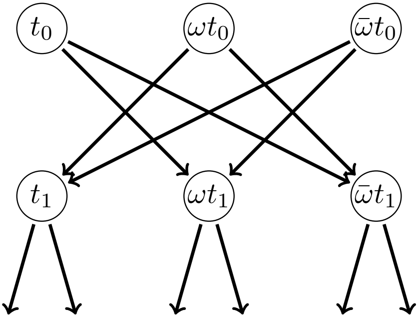


- t-invariants over $j = 12^3$ fall in two orbits of points, $\{2, 2\omega, 2\omega^2\}$ of multiplicity 2, and $\{-4, -4\omega, -4\omega^2\}$ of multiplicity 1.
- These points at the surface are linked by a 2-isogeny and to 2-depth 1, to $t = 8$.
- $\Psi_2(\omega x, \omega^2 y) = \omega\Psi_2(x, y)$: the choice of representative in the orbit gives rise to one of three distinct components of the 2-isogeny graph.

Initialization: $(t_0, t_1, t_2, \dots) = (2, 8, 8c, \dots)$ where c is a root of $x^2 - 8x - 2$.
Extension: random selection of a root t_{i+1} of $\Psi_2(t_i, x)$.

The full 2-isogeny graph has ascending edges from the depth one to $t_0 = 2$
If an isogeny is descending its only extension to a 2-isogeny chain is descending

WEBER INITIALIZATIONS - DISCRIMINANT -3

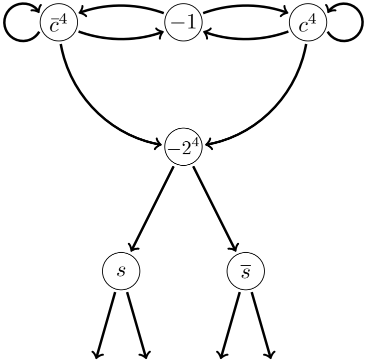


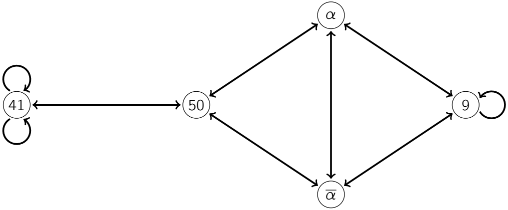
- $t_0 = -(\sqrt[3]{2})^4 = -2\sqrt[3]{2}$.
- $\{t_0, t_0\omega, t_0\omega^2\}$ are t-values over $j = 0$, each of multiplicity 3
- $t_1 = -t_0^2$, and $\Psi_2(t_0, t_1\omega) = \Psi_2(t_0, t_1\omega^2) = 0$,

Since 2 is inert, every path from t_0 is descending, so we may initialize the 2-isogeny chain with $(t_0, t_1\omega)$.

There are additional t-invariants at each depth > 0 which admit ascending and descending isogenies.
Any descending isogenies must rejoin this graph of descending isogenies from the surface.

WEBER INITIALIZATIONS - COVERING GRAPHS





THANK YOU FOR YOUR ATTENTION