ORIENTING SUPERSINGULAR ISOGENY GRAPHS

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Number-Theoretic Methods in Cryptology 2019
Sorbonne Université, Institut de Mathématiques de Jussieu
Paris, 26 June 2019

Isogeny Graphs

Definition

Given an elliptic curve E over k, and a finite set of primes S, we can associate an isogeny graph $\Gamma = (E, S)$

- whose vertices are elliptic curves isogenous to E over $\bar{k}$, and
- whose edges are isogenies of degree $\ell \in S$.

The vertices are defined up to $\bar{k}$-isomorphism (therefore represented by j-invariants), and the edges from a given vertex are defined up to a $\bar{k}$-isomorphism of the codomain.

If $S = \{\ell\}$, then we call Γ an ℓ-isogeny graph.

For an elliptic curve E/k and prime $\ell \neq \texttt{char}(k)$, the full ℓ-torsion subgroup is a 2-dimensional $\mathbb{F}_\ell$-vector space. Consequently, the set of cyclic subgroups is in bijection with $\mathbb{P}^1(\mathbb{F}_\ell)$, which in turn are in bijection with the set of ℓ-isogenies from E.

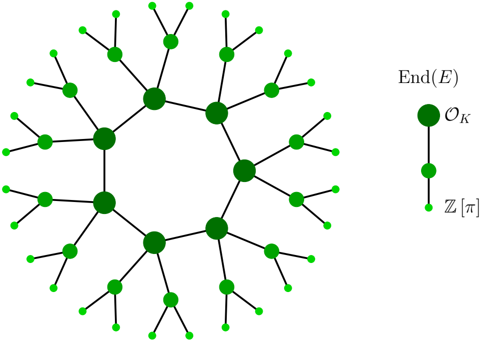
Thus the ℓ-isogeny graph of E is $(\ell + 1)$-regular (as a directed multigraph). In characteristic 0, if $\mathrm{End}(E) = \mathbb{Z}$, then this graph is a tree.

ORDINARY ISOGENY GRAPHS: VOLCANOES

Let End$(E) = \mathcal{O} \subseteq K$. The class group Cl$(\mathcal{O})$ (finite abelian group) acts faithfully and transitively on the set of elliptic curves with endomorphism ring $\mathcal{O}$:

$$E \longrightarrow E/E[\mathfrak{a}] \qquad E[\mathfrak{a}] = \{P \in E \mid \alpha(P) = 0 \ \forall \alpha \in \mathfrak{a}\}$$

Thus, the CM isogeny graphs can be modelled by an equivalent category of fractional ideals of K.

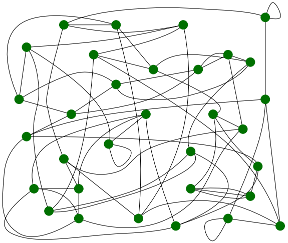


SUPERSINGULAR ISOGENY GRAPHS

The supersingular isogeny graphs are remarkable because the vertex sets are finite : there are $[p/12] + \epsilon_p$ curves. Moreover

- every supersingular elliptic curve can be defined over $\mathbb{F}_{p^2}$;
- all ℓ-isogenies are defined over $\mathbb{F}_{p^2}$;
- every endomorphism of E is defined over $\mathbb{F}_{p^2}$.

The lack of a commutative group acting on the set of supersingular elliptic curves/$\mathbb{F}_{p^2}$ makes the isogeny graph more complicated.



For this reason, supersingular isogeny graphs have been proposed for

- cryptographic hash functions (Goren–Lauter),
- post-quantum SIDH key exchange protocol.

MOTIVATING OSIDH

A new key exchange protocol, CSIDH, analogous to SIDH, uses only $\mathbb{F}_p$-rational elliptic curves (up to $\mathbb{F}_p$-isomorphism), and $\mathbb{F}_p$-rational isogenies.

The constraint to $\mathbb{F}_p$-rational isogenies can be interpreted as an orientation of the supersingular graph by the subring $\mathbb{Z}[\pi]$ of $\mathbf{End}(E)$ generated by the Frobenius endomorphism π.

We introduce a general notion of orienting supersingular elliptic curves.

Motivation

- Generalize CSIDH.
- Key space of SIDH: in order to have the two key spaces of similar size, we need to take $\ell_A^{e_A} \approx \ell_B^{e_B} \approx \sqrt{p}$. This implies that the space of choices for the secret key is limited to a fraction of the whole set of supersingular j-invariants over $\mathbb{F}_{p^2}$.
- A feature shared by SIDH and CSIDH is that the isogenies are constructed as quotients of rational torsion subgroups. The need for rational points limits the choice of the prime p

ORIENTATIONS

Let $\mathcal{O}$ be an order in an imaginary quadratic field. An *$\mathcal{O}$-orientation* on a supersingular elliptic curve E is an inclusion $\iota : \mathcal{O} \hookrightarrow \mathrm{End}(E)$, and a *$K$-orientation* is an inclusion $\iota : K \hookrightarrow \mathrm{End}^0(E) = \mathrm{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. An $\mathcal{O}$-orientation is *primitive* if $\mathcal{O} \simeq \mathrm{End}(E) \cap \iota(K)$.

Theorem

The category of K-oriented supersingular elliptic curves (E, ι), whose morphisms are isogenies commuting with the K-orientations, is equivalent to the category of elliptic curves with CM by K.

Let $\phi : E \to F$ be an isogeny of degree ℓ. A K-orientation $\iota : K \hookrightarrow \mathrm{End}^0(E)$ determines a K-orientation $\phi_*(\iota) : K \hookrightarrow \mathrm{End}^0(F)$ on F, defined by

$$\phi_*(\iota)(\alpha) = \frac{1}{\ell}\, \phi \circ \iota(\alpha) \circ \hat{\phi}.$$

CLASS GROUP ACTION

- $\mathsf{SS}(p) = \{$supersingular elliptic curves over $\overline{\mathbb{F}}_p$ up to isomorphism$\}$.
- $\mathsf{SS}_{\mathcal{O}}(p) = \{\mathcal{O}$-oriented s.s. elliptic curves over $\overline{\mathbb{F}}_p$ up to K-isomorphism$\}$.
- $\mathsf{SS}_{\mathcal{O}}^{pr}(p)$ =subset of primitive $\mathcal{O}$-oriented curves.

The set $\mathsf{SS}_{\mathcal{O}}(p)$ admits a transitive group action:

$$\mathcal{C}\ell(\mathcal{O}) \times \mathsf{SS}_{\mathcal{O}}(p) \longrightarrow \mathsf{SS}_{\mathcal{O}}(p) \qquad ([\mathfrak{a}], E) \longmapsto [\mathfrak{a}] \cdot E = E/E[\mathfrak{a}]$$

Proposition

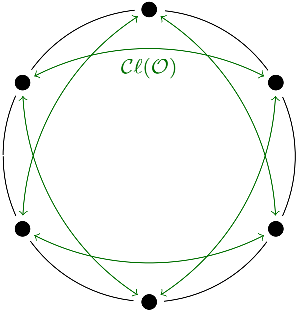
The class group $\mathcal{C}\ell(\mathcal{O})$ acts faithfully and transitively on the set of $\mathcal{O}$-isomorphism classes of primitive $\mathcal{O}$-oriented elliptic curves.

In particular, for fixed primitive $\mathcal{O}$-oriented E, we obtain a bijection of sets:

$$\mathcal{C}\ell(\mathcal{O}) \longrightarrow \mathsf{SS}_{\mathcal{O}}^{pr}(p) \qquad [\mathfrak{a}] \longmapsto [\mathfrak{a}] \cdot E$$

Vortex

We define a vortex to be the ℓ-isogeny subgraph whose vertices are isomorphism classes of $\mathcal{O}$-oriented elliptic curves with ℓ-maximal endomorphism ring, equipped with an action of $\mathcal{C}\ell(\mathcal{O})$.

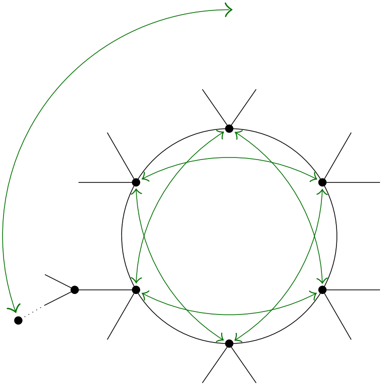


Instead of considering the union of different isogeny graphs, we focus on one single crater and we think of all the other primes as acting on it: the resulting object is a single isogeny circle rotating under the action of $\mathcal{C}\ell(\mathcal{O})$.

WHIRLPOOL

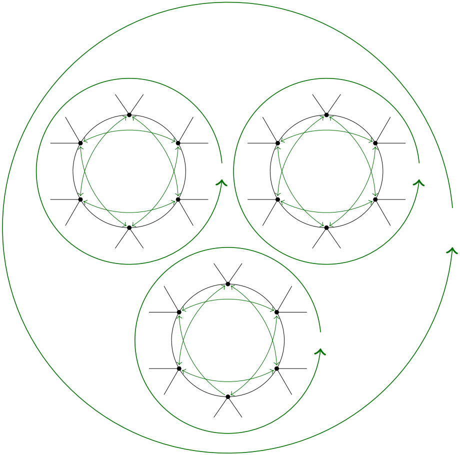
The action of $\mathcal{C}\ell(\mathcal{O})$ extends to the union $\bigcup_i SS_{\mathcal{O}_i}(p)$ over all superorders $\mathcal{O}_i$ containing $\mathcal{O}$ via the surjections $\mathcal{C}\ell(\mathcal{O}) \to \mathcal{C}\ell(\mathcal{O}_i)$.

We define a *whirlpool* to be a complete isogeny volcano acted on by the class group. We would like to think at isogeny graphs as moving objects.



WHIRLPOOL

Actually, we would like to take the ℓ-isogeny graph on the full $\mathcal{C}\ell(\mathcal{O}_K)$-orbit. This might be composed of several ℓ-isogeny orbits (craters), although the class group is transitive.



ISOGENY CHAINS

Definition

An ℓ-isogeny chain of length n from E_0 to E is a sequence of isogenies of degree ℓ:

$$E_0 \xrightarrow{\phi_0} E_1 \xrightarrow{\phi_1} E_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{n-1}} E_n = E.$$

The ℓ-isogeny chain is without backtracking if $\mathtt{ker}\,(\phi_{i+1} \circ \phi_i) \neq E_i[\ell],\ \forall i.$ The isogeny chain is descending (or ascending, or horizontal) if each ϕ_i is descending (or ascending, or horizontal, respectively).

Suppose that (E_i, ϕ_i) is a descending ℓ-isogeny chain with

$$\mathcal{O}_K \subseteq \mathsf{End}(E_0), \dots, \mathcal{O}_n = \mathbb{Z} + \ell^n \mathcal{O}_K \subseteq \mathsf{End}(E_n)$$

If $\mathfrak{q}$ is a split prime in $\mathcal{O}_K$ over $q \neq \ell, p$, and then the isogeny $\psi_0 : E_0 \to F_0 = E_0/E_0\,[\mathfrak{q}]$, can be extended to the ℓ-isogeny chain by pushing forward the cyclic group $C_0 = E_0\,[\mathfrak{q}]$:

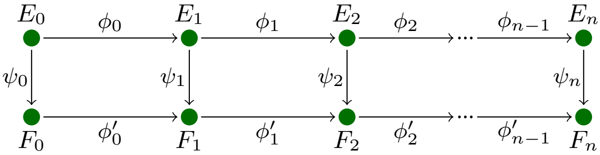
$$C_0 = E_0\,[\mathfrak{q}]\,,\ C_1 = \phi_0(C_0),\ \dots\ ,\ C_n = \phi_{n-1}(C_{n-1})$$

and defining $F_i = E_i/C_i$.

LADDERS

Definition

An ℓ-ladder of length n and degree q is a commutative diagram of ℓ-isogeny chains (E_i, ϕ_i), (F_i, ϕ_i') of length n connected by q-isogenies $\psi_i : E_i \to F_i$



We also refer to an ℓ-ladder of degree q as a q-isogeny of ℓ-isogeny chains.

We say that an ℓ-ladder is ascending (or descending, or horizontal) if the ℓ-isogeny chain (E_i, ϕ_i) is ascending (or descending, or horizontal, respectively).

We say that the ℓ-ladder is level if ψ_0 is a horizontal q-isogeny. If the ℓ-ladder is descending (or ascending), then we refer to the length of the ladder as its depth (or, respectively, as its height).

EFFECTIVE ENDOMORPHISM RINGS AND ISOGENIES

We say that a subring of $\mathrm{End}(E)$ is effective if we have explicit polynomials or rational functions which represent its generators.

Examples. $\mathbb{Z}$ in $\mathrm{End}(E)$ is effective. Effective imaginary quadratic subrings $\mathcal{O} \subset \mathrm{End}(E)$, are the subrings $\mathcal{O} = \mathbb{Z}[\pi]$ generated by Frobenius

In the Couveignes-Rostovtsev-Stolbunov constructions, or in the CSIDH protocol, one works with $\mathcal{O} = \mathbb{Z}[\pi]$.

- For large finite fields, the class group of $\mathcal{O}$ is large and the primes $\mathfrak{q}$ in $\mathcal{O}$ have no small generators.
  Factoring the division polynomial $\psi_q(x)$ to find the kernel polynomial of degree $(q-1)/2$ for $E[\mathfrak{q}]$ becomes relatively expensive.
- In SIDH, the ordinary protocol of De Feo, Smith, and Kieffer, or CSIDH, the curves are chosen such that the points of $E[\mathfrak{q}]$ are defined over a small degree extension κ/k, and working with rational points in $E(\kappa)$.
- We propose the use of an effective CM order $\mathcal{O}_K$ of class number 1.
  The kernel polynomial can be computed directly without need for a splitting field for $E[\mathfrak{q}]$, and the computation of a generator isogeny is a one-time precomputation.

MODULAR APPROACH

The use of modular curves for efficient computation of isogenies has an established history (see Elkies)

Modular Curve

The modular curve $\mathtt{X}(1) \simeq \mathbb{P}^1$ classifies elliptic curves up to isomorphism, and the function j generates its function field.

The modular polynomial $\Phi_m(X,Y)$ defines a correspondence in $\mathtt{X}(1) \times \mathtt{X}(1)$ such that $\Phi_m(j(E), j(E')) = 0$ if and only if there exists a cyclic m-isogeny ϕ from E to E', possibly over some extension field.

Definition

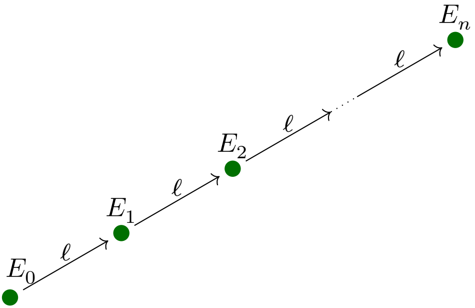
A *modular ℓ-isogeny chain* of length n over k is a finite sequence $(j_0, j_1, \dots, j_n)$ in k such that $\Phi_\ell(j_i, j_{i+1}) = 0$ for $0 \le i < n$.
A *modular ℓ-ladder* of length n and degree q over k is a pair of modular ℓ-isogeny chains

$$(j_0, j_1, \dots, j_n) \text{ and } (j'_0, j'_1, \dots, j'_n),$$

such that $\Phi_q(j_i, j'_i) = 0$.

OSIDH - Introduction

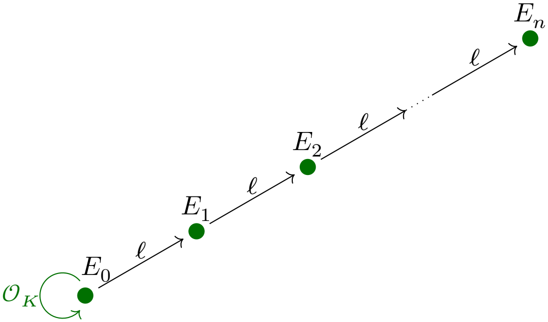
We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.



OSIDH - Introduction

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

- For $\ell = 2$ (or 3) a suitable candidate for $\mathcal{O}_K$ could be the Gaussian integers $\mathbb{Z}[i]$ or the Eisenstein integers $\mathbb{Z}[\omega]$.



OSIDH - INTRODUCTION

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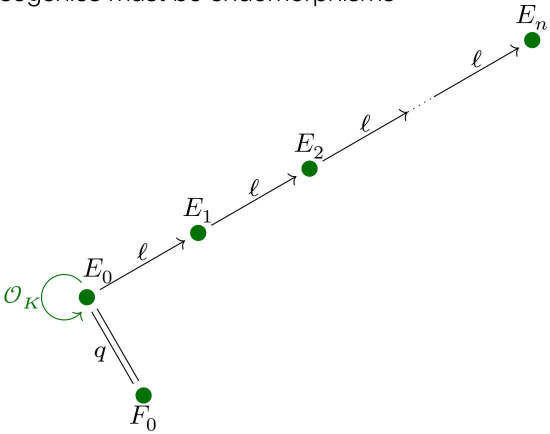
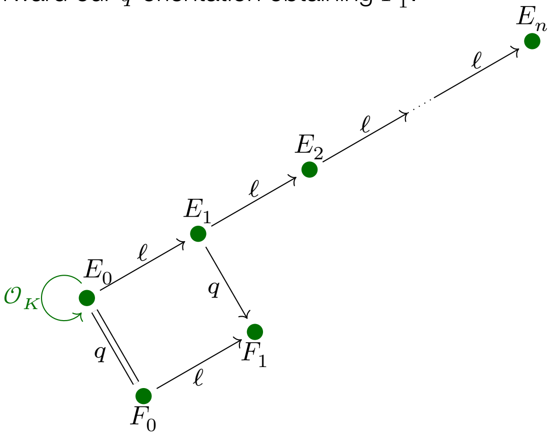
- Horizontal isogenies must be endomorphisms



OSIDH - Introduction

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

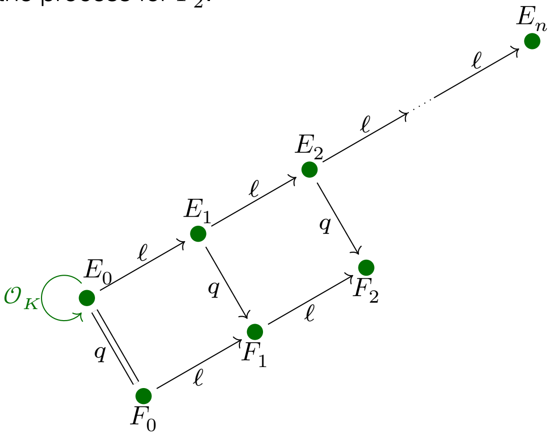
- We push forward our q-orientation obtaining F_1.



OSIDH - INTRODUCTION

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

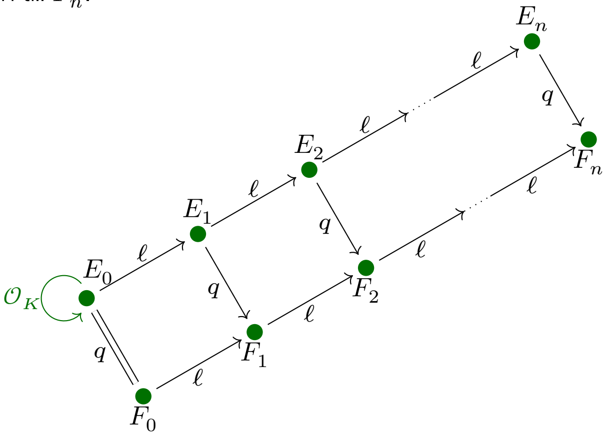
- We repeat the process for F_2.



OSIDH - INTRODUCTION

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

- And again till F_n.



HOW FAR SHOULD WE GO?

In order to have the action of $\mathcal{C}\ell(\mathcal{O})$ cover a large portion of the supersingular elliptic curves, we require $\ell^n \sim p$, i.e., $n \sim \texttt{log}_\ell(p)$.

- $\#SS^{pr}_{\mathcal{O}}(p) = h(\mathcal{O}_n) =$class number of $\mathcal{O}_n = \mathbb{Z} + \ell^n \mathcal{O}_K$.
- Class Number Formula

$$h(\mathbb{Z} + m\mathcal{O}_K) = \frac{h(\mathcal{O}_K)m}{[\mathcal{O}_K^\times : \mathcal{O}^\times]} \prod_{p|m} \left(1 - \left(\frac{\Delta_K}{p}\right)\frac{1}{p}\right)$$

- Units

$$\mathcal{O}_K^\times = \begin{cases} \{\pm 1\} & \text{if } \Delta_K < -4 \\ \{\pm 1, \pm i\} & \text{if } \Delta_K = -4 \\ \{\pm 1, \pm \omega, \pm \omega^2\} & \text{if } \Delta_K = -3 \end{cases} \Rightarrow [\mathcal{O}_K^\times : \mathcal{O}^\times] = \begin{cases} 1 & \text{if } \Delta_K < -4 \\ 2 & \text{if } \Delta_K = -4 \\ 3 & \text{if } \Delta_K = -3 \end{cases}$$

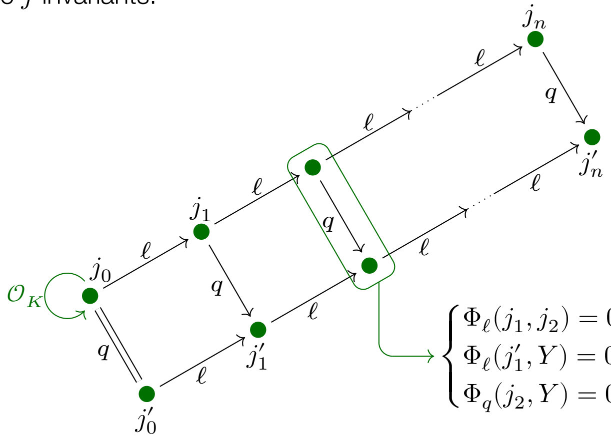
- Number of Supersingular curves

$$\#\text{SS}(p) = \left[\frac{p}{12}\right] + \epsilon_p \qquad \epsilon_p \in \{0, 1, 2\}$$

Therefore, $h(\ell^n \mathcal{O}_K) = \dfrac{1 \cdot \ell^n}{2 \text{ or } 3}\left(1 - \left(\dfrac{\Delta_K}{\ell}\right)\dfrac{1}{\ell}\right) = \left[\dfrac{p}{12}\right] + \epsilon_p \implies p \sim \ell^n$

OSIDH - Introduction & Modular Approach

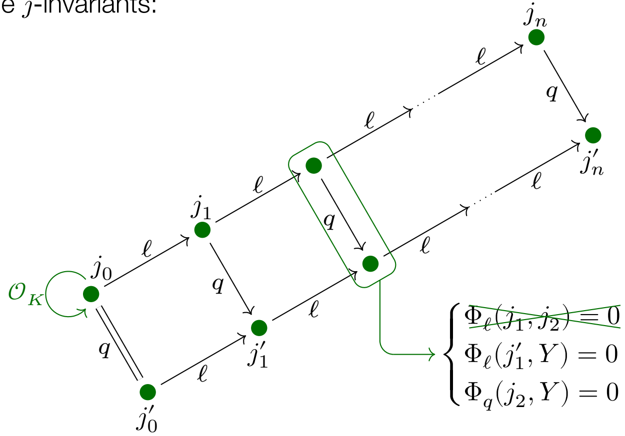
If we look at modular polynomials $\Phi_\ell(X,Y)$ and $\Phi_q(X,Y)$ we realize that all we need are the j-invariants:



$$\begin{cases} \Phi_\ell(j_1, j_2) = 0 \\ \Phi_\ell(j'_1, Y) = 0 \\ \Phi_q(j_2, Y) = 0 \end{cases}$$

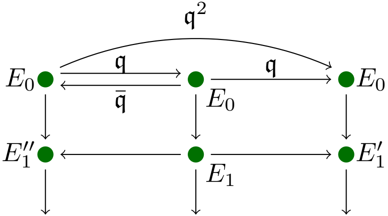
OSIDH - Introduction & Modular Approach

If we look at modular polynomials $\Phi_\ell(X,Y)$ and $\Phi_q(X,Y)$ we realize that all we need are the j-invariants:



Since j_2 is given (the initial chain is known) and supposing that j'_1 has already been constructed, j'_2 is determined by a system of two equations

HOW MANY STEPS BEFORE THE IDEALS ACT DIFFERENTLY?



$E_i' \neq E_i''$ if and only if $\mathfrak{q}^2 \cap \mathcal{O}_i$ is not principal and the probability that a random ideal in $\mathcal{O}_i$ is principal is $1/h(\mathcal{O}_i)$. In fact, we can do better; we write $\mathcal{O}_K = \mathbb{Z}[\omega]$ and we observe that if $\mathfrak{q}^2$ was principal, then

$$q^2 = \mathsf{N}(\mathfrak{q}^2) = \mathsf{N}(a + b\ell^i\omega)$$

since it would be generated by an element of $\mathcal{O}_i = \mathbb{Z} + \ell^i\mathcal{O}_K$. Now

$$\mathsf{N}(a + b\ell^i) = a^2 \pm abt\ell^i + b^2 s\ell^{2i} \quad \text{where} \quad \omega^2 + t\omega + s = 0$$

Thus, as soon as $\ell^{2i} \gg q^2$, we are guaranteed that $\mathfrak{q}^2$ is not principal.

A FIRST NAIVE PROTOCOL

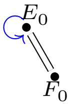
PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to ... \to E_n$		
	ALICE	**BOB**

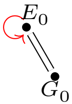
A FIRST NAIVE PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \ldots \to E_n$

ALICE **BOB**

Choose a primitive $\mathcal{O}_K$-orientation of E_0





A FIRST NAIVE PROTOCOL

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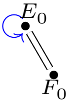
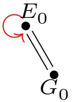
	ALICE	**BOB**
Choose a primitive $\mathcal{O}_K$-orientation of E_0		
Push it forward to depth n	$\underbrace{E_0 = F_0 \to F_1 \to \ldots \to F_n}_{\phi_A}$	$\underbrace{E_0 = G_0 \to G_1 \to \ldots \to G_n}_{\phi_B}$

A FIRST NAIVE PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \dots \to E_n$

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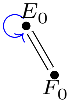
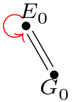
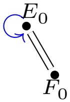
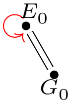
Choose a primitive $\mathcal{O}_K$-orientation of E_0		
Push it forward to depth n	$\underbrace{E_0 = F_0 \to F_1 \to \dots \to F_n}_{\phi_A}$	$\underbrace{E_0 = G_0 \to G_1 \to \dots \to G_n}_{\phi_B}$
Exchange data	$\{G_i\}_{i=1}^n$	$\{F_i\}_{i=1}^n$

A FIRST NAIVE PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \ldots \to E_n$

	ALICE	**BOB**
Choose a primitive $\mathcal{O}_K$-orientation of E_0		
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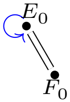
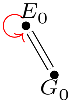
Exchange data	$\{G_i\}_{i=1}^n$	$\{F_i\}_{i=1}^n$
Compute shared secret	Compute $\phi_A \cdot \{G_i\}$	Compute $\phi_B \cdot \{F_i\}$

A FIRST NAIVE PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \dots \to E_n$

	ALICE	**BOB**
Choose a primitive $\mathcal{O}_K$-orientation of E_0		
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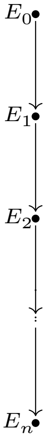
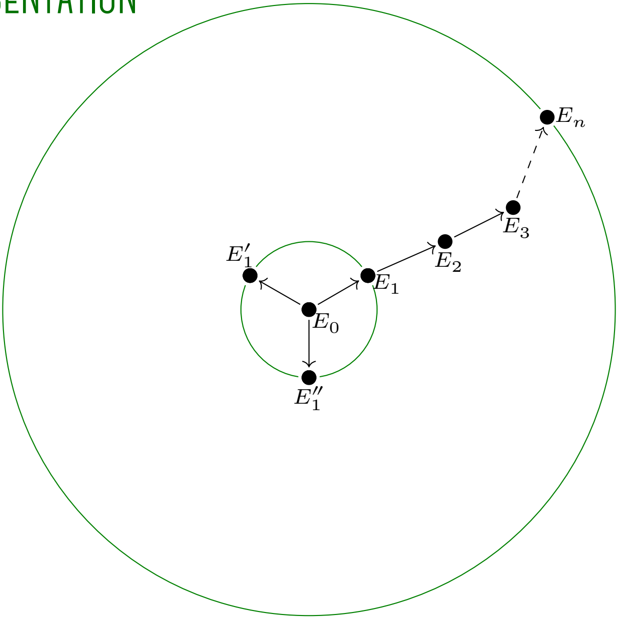
Exchange data	$\{G_i\}_{i=1}^n$	$\{F_i\}_{i=1}^n$
Compute shared secret	Compute $\phi_A \cdot \{G_i\}$	Compute $\phi_B \cdot \{F_i\}$

In the end, Alice and Bob will share a new chain $E_0 \to H_1 \to \dots \to H_n$

GRAPHIC REPRESENTATION





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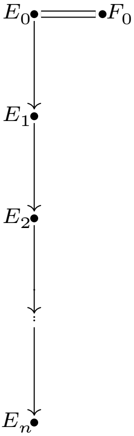

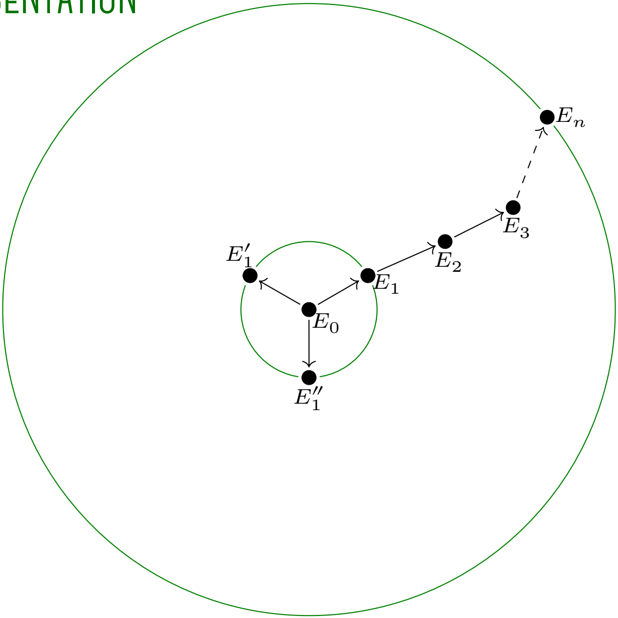

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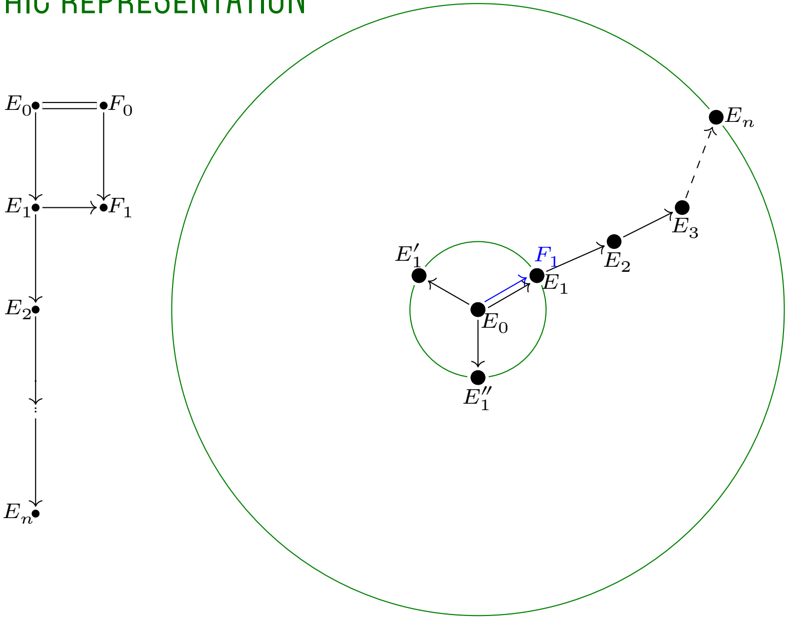

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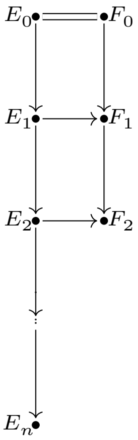

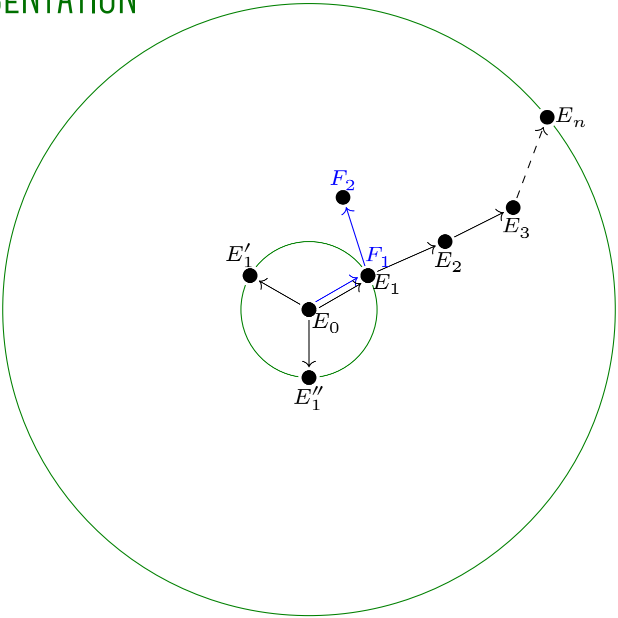

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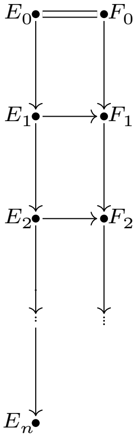

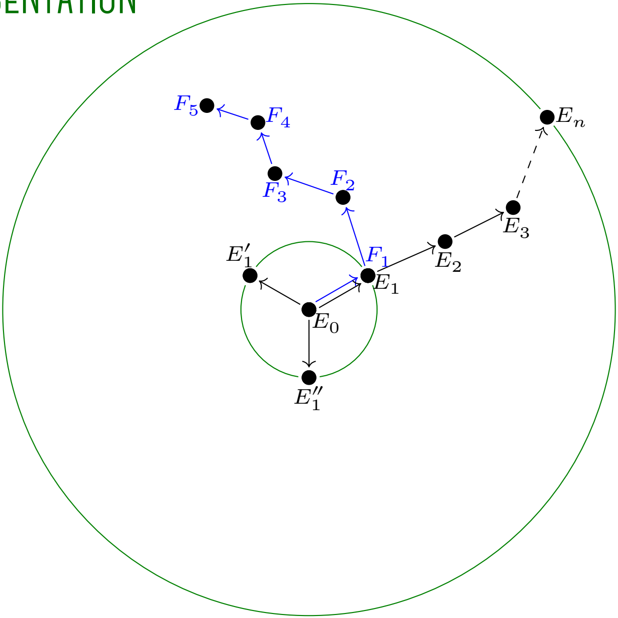

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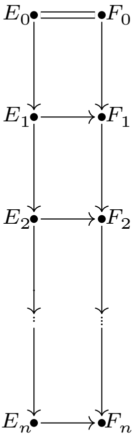

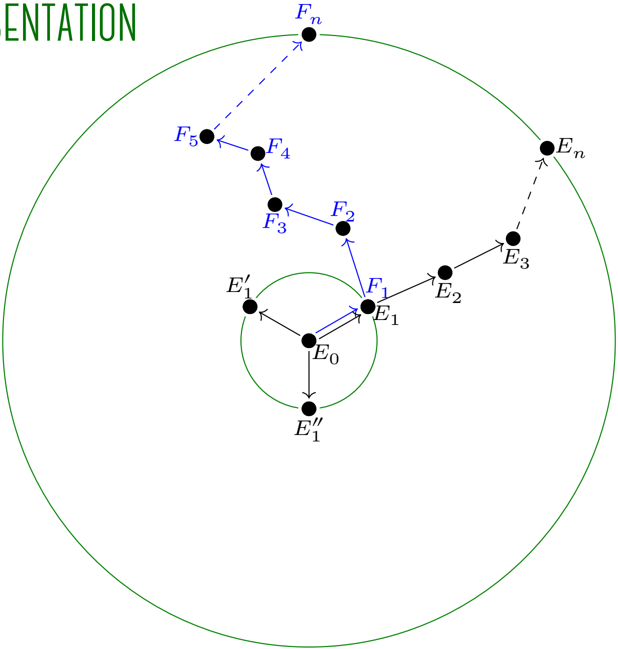

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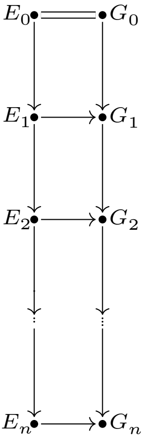

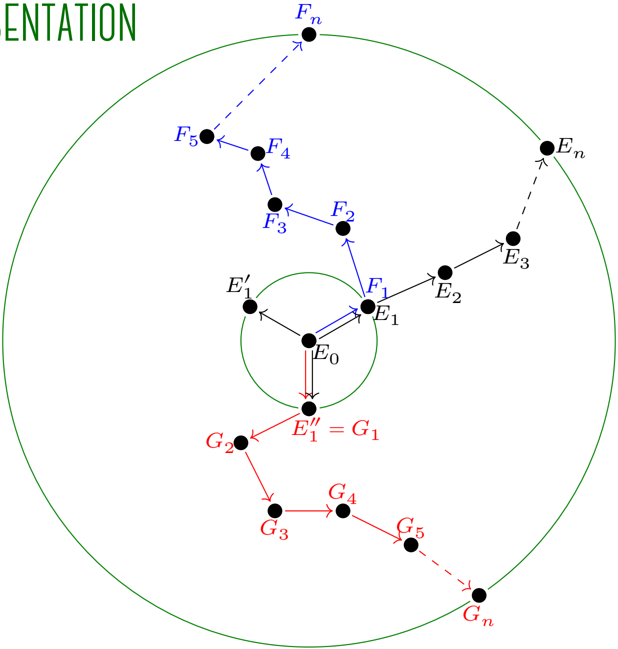

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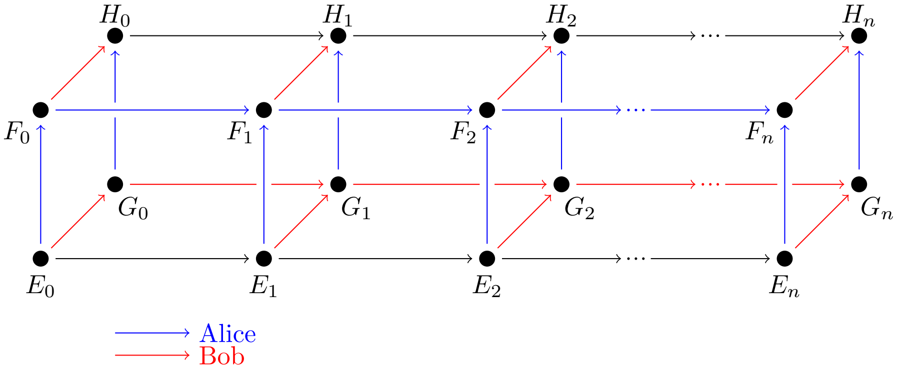


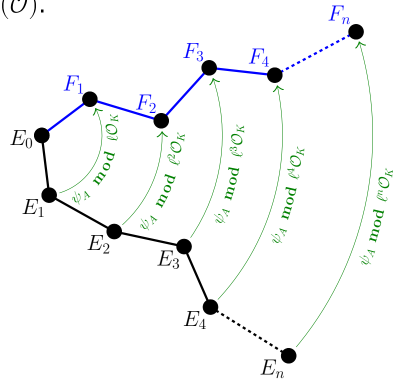
A FIRST NAIVE PROTOCOL - WEAKNESS

In reality, sharing (F_i) and (G_i) reveals too much of the private data.

From the short exact sequence of class groups:

$$1 \to \frac{(\mathcal{O}_K/\ell^n\mathcal{O}_K)^\times}{\mathcal{O}_K^\times(\mathbb{Z}/\ell^n\mathbb{Z})^\times} \to \mathcal{C}\ell(\mathcal{O}) \to \mathcal{C}\ell(\mathcal{O}_K) \to 1$$

an adversary can compute successive approximations (mod ℓ^i) to ϕ_A and ϕ_B modulo ℓ^n hence in $\mathcal{C}\ell(\mathcal{O})$.



OSIDH PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \dots \to E_n$ and a set of splitting primes $\mathfrak{p}_1, \dots, \mathfrak{p}_t \subseteq \mathcal{O}_n \subseteq \mathrm{End}(E_n) \cap K \subseteq \mathcal{O}_K$

ALICE	BOB

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	ALICE	**BOB**
Choose integers in a bound $[-r, r]$	$(e_1, \dots, e_t)$	$(d_1, \dots, d_t)$

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	ALICE	**BOB**
Choose integers in a bound $[-r, r]$	$(e_1, \dots, e_t)$	$(d_1, \dots, d_t)$
Construct an isogenous curve	$F_n = E_n / E_n \left[\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}\right]$	$G_n = E_n / E_n \left[\mathfrak{p}_1^{d_1} \cdots \mathfrak{p}_t^{d_t}\right]$

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Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$

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Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$
... and their conjugates	$F_n \to F_{n,i}^{(1)} \to \dots \to F_{n,i}^{(r-1)} \to F_{n,1}^{(r)}$	$G_n \to G_{n,i}^{(1)} \to \dots \to G_{n,i}^{(r-1)} \to G_{n,1}^{(r)}$

OSIDH PROTOCOL

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Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$
... and their conjugates	$F_n \to F_{n,i}^{(1)} \to \dots \to F_{n,i}^{(r-1)} \to F_{n,1}^{(r)}$	$G_n \to G_{n,i}^{(1)} \to \dots \to G_{n,i}^{(r-1)} \to G_{n,1}^{(r)}$
Exchange data	G_n+directions	F_n+directions

OSIDH PROTOCOL

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Precompute all directions $\forall i$	$F^{(-r)}_{n,i} \leftarrow F^{(-r+1)}_{n,i} \leftarrow \dots \leftarrow F^{(1)}_{n,i} \leftarrow F_n$	$G^{(-r)}_{n,i} \leftarrow G^{(-r+1)}_{n,i} \leftarrow \dots \leftarrow G^{(1)}_{n,i} \leftarrow G_n$
... and their conjugates	$F_n \to F^{(1)}_{n,i} \to \dots \to F^{(r-1)}_{n,i} \to F^{(r)}_{n,1}$	$G_n \to G^{(1)}_{n,i} \to \dots \to G^{(r-1)}_{n,i} \to G^{(r)}_{n,1}$
Exchange data	G_n+directions	F_n+directions
Compute shared data	Takes e_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.	Takes d_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.

OSIDH PROTOCOL

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Construct an isogenous curve	$F_n = E_n / E_n \left[\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}\right]$	$G_n = E_n / E_n \left[\mathfrak{p}_1^{d_1} \cdots \mathfrak{p}_t^{d_t}\right]$
Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$
... and their conjugates	$F_n \to F_{n,i}^{(1)} \to \dots \to F_{n,i}^{(r-1)} \to F_{n,1}^{(r)}$	$G_n \to G_{n,i}^{(1)} \to \dots \to G_{n,i}^{(r-1)} \to G_{n,1}^{(r)}$
Exchange data	G_n+directions	F_n+directions
Compute shared data	Takes e_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.	Takes d_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.

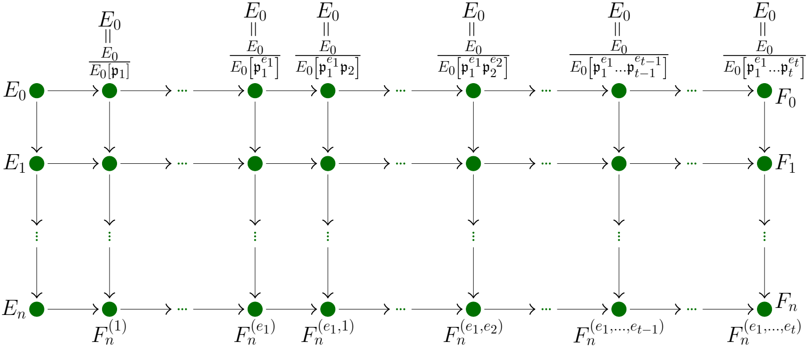
In the end, they share $H_n = E_n / E_n \left[\mathfrak{p}_1^{e_1+d_1} \cdot \ldots \cdot \mathfrak{p}_t^{e_t+d_t}\right]$

OSIDH PROTOCOL - GRAPHIC REPRESENTATION I

The first step consists of choosing the secret keys; these are represented by a sequence of integers $(e_1, \dots, e_t)$ such that $|e_i| \leq r$. The bound r is taken so that the number $(2r+1)^t$ of curves that can be reached is sufficiently large. This choice of integers enables Alice to compute a new elliptic curve

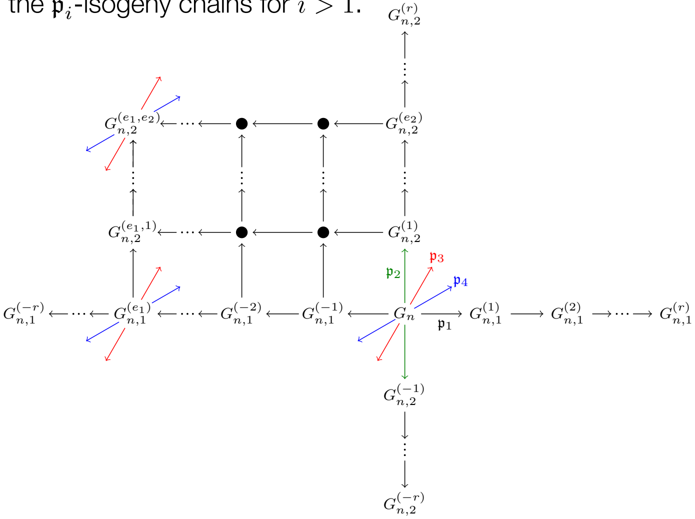
$$F_n = \frac{E_n}{E_n\left[\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}\right]}$$

by means of constructing the following commutative diagram



OSIDH PROTOCOL - GRAPHIC REPRESENTATION II

Once that Alice obtain from Bob the curve G_n together with the collection of data encoding the directions, she takes e_1 steps in the $\mathfrak{p}_1$-isogeny chain and push forward all the $\mathfrak{p}_i$-isogeny chains for $i > 1$.



HARD PROBLEMS

Endomorphism ring problem

Given a supersingular elliptic curve $E/\mathbb{F}_{p^2}$ and $\pi = [p]$, determine **End**(E) as an abstract ring or an explicit basis for it over $\mathbb{Z}$ (or for **End**$^0(E)$ over $\mathbb{Q}$).

Endomorphism Generators Problem

Given a supersingular elliptic curve $E/\mathbb{F}_{p^2}$, $\pi = [p]$, an imaginary quadratic order $\mathcal{O}$ admitting an embedding in **End**(E) and a collection of compatible $(\mathcal{O}, \mathfrak{q}^n)$-orientations of E for $(\mathfrak{q}, n) \in S$, determine

1. An explicit endomorphism $\phi \in \mathcal{O} \subseteq$ **End**(E)
2. A generator ϕ of $\mathcal{O} \subseteq$ **End**(E)

Suppose $S = \{(\mathfrak{q}, n)\} = \{(\mathfrak{q}_1, n_1), \dots, (\mathfrak{q}_t, n_t)\}$ where $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ are pairwise distinct primes such that

$$\begin{aligned}[0, \dots, n_1] \times \dots \times [0, \dots, n_t] &\longrightarrow \mathcal{C}\ell(\mathcal{O}) \\ (e_1, \dots, e_t) &\longrightarrow [\mathfrak{q}_1^{e_1} \cdot \dots \cdot \mathfrak{q}_t^{e_t}]\end{aligned}$$

is injective. Then, the problem should remain difficult.

Security parameters - first choice

Consider an arbitrary supersingular endomorphism ring $\mathcal{O}_{\mathfrak{B}} \subset \mathfrak{B}$ with discriminant p^2. There is a positive definite rank 3 quadratic form

$$\begin{array}{rcl} \text{disc} : \mathcal{O}_{\mathfrak{B}}/\mathbb{Z} & \longrightarrow & \mathbb{Z} \\ /\!/ \quad \alpha & \longmapsto & |\text{disc}(\alpha)| = |\text{disc}\left(\mathbb{Z}\left[\alpha\right]\right)| \\ \bigwedge^2\left(\mathcal{O}_{\mathfrak{B}}\right) \supseteq \mathbb{Z} \wedge \mathcal{O}_{\mathfrak{B}} & & \end{array}$$

representing discriminants of orders embedding in $\mathcal{O}_{\mathfrak{B}}$.

The general order $\mathcal{O}_{\mathfrak{B}}$ has a reduced basis $1 \wedge \alpha_1, 1 \wedge \alpha_2, 1 \wedge \alpha_3$ satisfying

$$|\text{disc}(1 \wedge \alpha_i)| = \Delta_i \text{ where } \Delta_i \sim p^{2/3}$$

(Minkowski bound: $c_1 p^2 \leq \Delta_1 \Delta_2 \Delta_3 \leq c_2 p^2$).

In order to hide $\mathcal{O}_n$ in $\mathcal{O}_{\mathfrak{B}}$ we impose

$$\ell^{2n}|\Delta_K| > c p^{2/3} \qquad \Rightarrow \qquad n \sim \frac{\log_\ell(p)}{3}$$

so that there is no special imaginary quadratic subring in $\mathcal{O}_{\mathfrak{B}} = \text{End}(E_n)$.

Future work:

- Security analysis and setting security parameters.
- Implementation and algorithmic optimization.
- Use of canonical liftings.

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THANK YOU FOR YOUR ATTENTION