ORIENTING SUPERSINGULAR ISOGENY GRAPHS

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ELLIPTIC CURVES

Let k be a field of characteristic $\neq 2, 3$. An elliptic curve E defined over k is a smooth projective curve of genus 1 defined by a Weierstrass equation

$$E : Y^2 Z = X^3 + aXZ^2 + bZ^3$$

where $a, b \in k$ are such that $4a^3 + 27b^2 \neq 0$.

In general we work with the affine equation of E, i.e., $E : y^2 = x^3 + ax + b$.

We distinguish the point $O = (0 : 1 : 0)$ (called *point at infinity*).

There is a way of adding points on E based on Bezout's theorem (we fix the point O and we define the sum of three co-linear points to be O). This law endows the set of k-rational points with a group structure where O plays the role of identity element. We write $E(k)$.

ISOMORPHISMS OF ELLIPTIC CURVES

An isomorphism of elliptic curves is an invertible morphism of algebraic curves. They are often referred to as *admissible (linear) change of variables*.

Isomorphisms

Invertible algebraic maps between elliptic curves are of the form

$$(x, y) \to (u^2 x, u^3 y) \qquad \text{for some } u \in \bar{k}.$$

Isomorphisms between elliptic curves are group isomorphisms.

Isomorphism classes are described by an invariant:

j-invariant

The j-invariant of an elliptic curve $E : y^2 = x^3 + ax + b$ is

$$j(E) = 1728 \frac{4a^3}{4a^3 + 27b^2}$$

Two elliptic curves E, E' are isomorphic over $\bar{k}$ if and only if $j(E) = j(E')$.

GROUP STRUCTURE

Let E be an elliptic curve defined over a field k and m an integer. The m-torsion subgroup of E is

$$E[m] = \{P \in E\left(\bar{k}\right) \mid mP = O\}$$

Torsion structure

Let E be an elliptic curve defined over an algebraic closed field $\bar{k}$ of characteristic p. If p does not divide m or $p = 0$, then

$$E[m] \simeq \frac{\mathbb{Z}}{m\mathbb{Z}} \times \frac{\mathbb{Z}}{m\mathbb{Z}}$$

If the $p > 0$, then

$$E[p^r] \simeq \begin{cases} \frac{\mathbb{Z}}{p^r\mathbb{Z}} & \text{Ordinary case} \\ \{O\} & \text{Supersingular case} \end{cases}$$

Isogenies

We want to study relationships between isomorphisms classes of elliptic curves.

Isogenies

An isogeny $\phi : E \to E'$ between two elliptic curves is

- A map $E \to E'$ such that $\phi(P + Q) = \phi(P) + \phi(Q)$.
- A surjective group morphisms (in the algebraic closure).
- A group morphism with finite kernel.
- A non-constant algebraic map of projective varieties such that $\phi(O_E) = O_{E'}$.
- An algebraic morphism given by rational maps

$$\phi(x, y) = \left(\frac{f_1(x, y)}{g_1(x, y)}, \frac{f_2(x, y)}{g_2(x, y)} \right)$$

The first example of isogeny is the multiplication by n map: $[n] : E \to E$.
If $k = \mathbb{F}_q$ we also have the Frobenius morphism $\pi : (x, y) \to (x^q, y^q)$.

ATTRIBUTES OF ISOGENIES

Let $\phi : E \to E'$ be an isogeny defined over a field k, $\text{char}(k) = p$. We define $k(E), k(E')$ to be the function fields of E and E'; by composing ϕ with elements of $k(E')$ we obtain a subfield $\phi^*(k(E'))$ of $k(E)$.

- The degree of ϕ is defined to be $\texttt{deg}\ \phi = [k(E) : \phi^* k(E')]$.
- ϕ is said separable, inseparable or purely inseparable if the corresponding extension of function fields is.
- If ϕ is separable then $\texttt{deg}\ \phi = \#\texttt{ker}\ \phi$ while in the purely inseparable case $\texttt{ker}\ \phi = \{O\}$ and $\texttt{deg}\ \phi = p^r$ some r.
- Given any isogeny $\phi : E \to E'$ there always exists a unique isogeny $\hat{\phi} : E' \to E$, called the *dual isogeny*, such that

$$\phi \circ \hat{\phi} = [\texttt{deg}\ \phi]_{E'} \qquad \hat{\phi} \circ \phi = [\texttt{deg}\ \phi]_{E}$$

THEOREMS ON ISOGENIES

Theorem

For every finite subgroup $G \subset E(\bar{k})$, there exist a unique (up to isomorphism) elliptic curve $E' = E/G$ and a unique separable isogeny $E \to E'$ of degree $\#G$. Further, any separable isogeny arises in this way.

Given G, Velu's formula enables one to find explicit description for ϕ.

Theorem (Tate)

Two elliptic curves E and E' defined over a finite field k are isogenous over k if and only if $\#E(k) = \#E'(k)$.

Observe that there exists an algorithm (Schoof - 1985) which, using isogenies, compute the cardinality of E in polynomial time.

ENDOMORPHISMS

An endomorphism of an elliptic curve E is an isogeny form E to itself.

Endomorphism ring

The endomorphism ring $\mathsf{End}(E) = \mathsf{End}_{\bar{k}}(E)$ of an elliptic curve E/k is the set of all endomorphisms of E (together with the 0-map) endowed with sum and multplication.

The endomorphism ring always contains a copy of $\mathbb{Z}$ in the form of the multiplication by m maps.
If k is a finite field we also have the Frobenius endomorphism.

Theorem (Hasse)

Let E be an elliptic curve defined over a finite field with q elements. Its Frobenius endomorphism satisfies a quadratic equation $\pi^2 - t\pi + q = 0$ for some $|t| \leq 2\sqrt{q}$, called the trace of π.

THEOREMS ON ENDOMORPHISMS

Let E be an elliptic curve defined over a finite field k. End(E) has dimension either 2 or 4 as a $\mathbb{Z}$-module.

Theorem (Deuring)

Let E/k be an elliptic curve over a finite field k of characteristic $p > 0$. End(E) is isomorphic to one of the following:

- An order $\mathcal{O}$ in a quadratic imaginary field; we say that E is ordinary.
- A maximal order in a quaternion algebra; we say that E is supersingular.

Isogenous curves are always either both ordinary, or both supersingular.

Theorem (Serre-Tate)

Two elliptic curves E_0 and E_1 defined over a finite field k are isogenous if and only if End$(E_0) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq$ End$(E_1) \otimes_{\mathbb{Z}} \mathbb{Q}$.

ISOGENY GRAPHS

Definition

Given an elliptic curve E over k, and a finite set of primes S, we can associate an isogeny graph $\Gamma = (E, S)$

- whose vertices are elliptic curves isogenous to E over $\bar{k}$, and
- whose edges are isogenies of degree $\ell \in S$.

The vertices are defined up to $\bar{k}$-isomorphism (therefore represented by j-invariants), and the edges from a given vertex are defined up to a $\bar{k}$-isomorphism of the codomain.

If $S = \{\ell\}$, then we call Γ an ℓ-isogeny graph.

For an elliptic curve E/k and prime $\ell \neq \mathtt{char}(k)$, the full ℓ-torsion subgroup is a 2-dimensional $\mathbb{F}_\ell$-vector space. Consequently, the set of cyclic subgroups is in bijection with $\mathbb{P}^1(\mathbb{F}_\ell)$, which in turn are in bijection with the set of ℓ-isogenies from E.

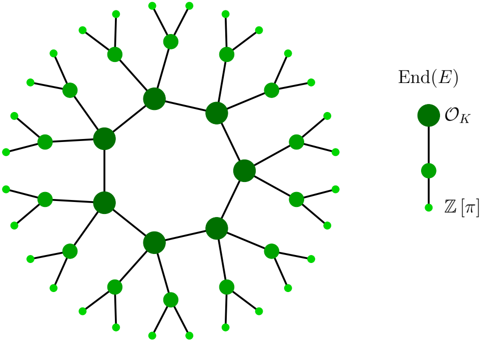
Thus the ℓ-isogeny graph of E is $(\ell + 1)$-regular (as a directed multigraph). In characteristic 0, if $\mathrm{End}(E) = \mathbb{Z}$, then this graph is a tree.

ORDINARY ISOGENY GRAPHS: VOLCANOES

Let $\mathsf{End}(E) = \mathcal{O} \subseteq K$. The class group $\mathsf{Cl}(\mathcal{O})$ (finite abelian group) acts faithfully and transitively on the set of elliptic curves with endomorphism ring $\mathcal{O}$:

$$E \longrightarrow E/E[\mathfrak{a}] \qquad E[\mathfrak{a}] = \{P \in E \mid \alpha(P) = 0 \ \forall \alpha \in \mathfrak{a}\}$$

Thus, the CM isogeny graphs can be modelled by an equivalent category of fractional ideals of K.



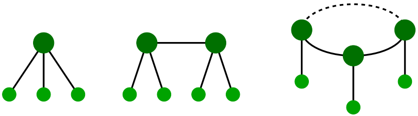
STRUCTURE OF VOLCANOES

Let E and E' be to elliptic curves with endomorphism rings $\mathcal{O}$ and $\mathcal{O}'$ respectively and let $\phi : E \to E'$ be an ℓ isogeny.

- If $\mathcal{O} = \mathcal{O}'$ we say that ϕ is horizontal;
- If $[\mathcal{O}' : \mathcal{O}] = \ell$ we say that ϕ is ascending;
- If $[\mathcal{O} : \mathcal{O}'] = \ell$ we say that ϕ is descending.

Crater

The crater consists of $h(\mathcal{O}_K) = \#\mathcal{C}\ell(\mathcal{O}_K)$ Elliptic curves. Depending on the behaviour of ℓ in $\mathcal{O}_K$ we can have one or multiple craters:



$\left(\frac{\Delta_K}{\ell}\right) = -1$ $\left(\frac{\Delta_K}{\ell}\right) = 0$ $\left(\frac{\Delta_K}{\ell}\right) = +1$

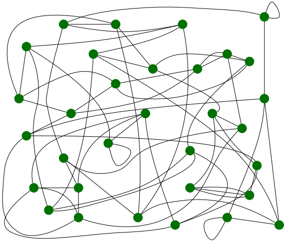
The height of the volcano is $\nu_\ell\left([\mathcal{O}_K : \mathbb{Z}[\pi]]\right)$.

SUPERSINGULAR ISOGENY GRAPHS

The supersingular isogeny graphs are remarkable because the vertex sets are finite : there are $(p+1)/12 + \epsilon_p$ curves. Moreover

- every supersingular elliptic curve can be defined over $\mathbb{F}_{p^2}$;
- all ℓ-isogenies are defined over $\mathbb{F}_{p^2}$;
- every endomorphism of E is defined over $\mathbb{F}_{p^2}$.

The lack of a commutative group acting on the set of supersingular elliptic curves/$\mathbb{F}_{p^2}$ makes the isogeny graph more complicated.



For this reason, supersingular isogeny graphs have been proposed for

- cryptographic hash functions (Goren–Lauter),
- post-quantum SIDH key exchange protocol.

SIDH - L. De Feo & D. Jao, 2011

Supersingular isogeny Diffie-Hellman

- Fix two small primes ℓ_A and ℓ_B;
- Choose a prime p such that $p+1 = \ell_A^a \ell_B^b f$ for a small correction term f;
- Pick a random supersingular elliptic curve $E/\mathbb{F}_{p^2}$: $E(\mathbb{F}_{p^2}) \simeq \left(\frac{\mathbb{Z}}{(p+1)\mathbb{Z}}\right)^2$
- Alice consider $E[\ell_A^a] = \langle P_A, Q_A \rangle$ while Bob takes $E[\ell_B^b] = \langle P_B, Q_B \rangle$.
- **Secret Data:** $R_A = m_A P_A + n_A Q_A$ and $R_B = m_B P_B + n_B Q_B$.
- **Private Key:** isogenies $\phi_A : E \to E_A = E/E\langle R_A \rangle$ and $\phi_B : E \to E_B = E/E\langle R_B \rangle$.
- **Shared Data:** E_A, $\phi_A(P_B)$, $\phi_A(Q_B)$ and E_B, $\phi_B(P_A)$, $\phi_B(Q_A)$.
- **Shared Key:** $E/E\langle R_A, R_B \rangle = E_B/\langle \phi_B(R_A) \rangle = E_A/\langle \phi_A(R_B) \rangle$.

CSIDH - W. Castryck & T. Lange & C. Martindale & L. Panny & J. Renes, 2018

It is an adaptation of the Couveignes–Rostovtsev–Stolbunov scheme to supersingular elliptic curves.

Commutative Supersingular isogeny Diffie-Hellman

- Fix a prime $p = 4 \cdot \ell_1 \cdot \ldots \cdot \ell_t - 1$ for small distinct odd primes ℓ_i.
- The elliptic curve $E_0 : y^2 = x^3 + x/\mathbb{F}_p$ is supersingular and its endomorphism ring restricted to $\mathbb{F}_p$ is $\mathcal{O} = \mathbb{Z}[\pi]$ (commutative).
- All Montgomery curves $E_A : y^2 = x^3 + Ax^2 + x/\mathbb{F}_p$ that are supersingular, appear in the $\mathcal{C}\ell(\mathcal{O})$-orbit of E_0 (easy to store data).
- **Private Key:** it is an n-tuple of integers $(e_1, \ldots, e_t)$ sampled in a range $\{-m, \ldots, m\}$ representing an ideal class $[\mathfrak{a}] = [\mathfrak{l}_1^{e_1} \cdot \ldots \cdot \mathfrak{l}_t^{e_t}] \in \mathcal{C}\ell(\mathcal{O})$ where $\mathfrak{l}_i = (\ell_i, \pi - 1)$.
- **Public Key:** The Montgomery coefficients A of the elliptic curve $E_A = [\mathfrak{a}] \cdot E_0 : y^2 = x^3 + Ax^2 + x$.
- **Shared Key:** If Alice and Bob have private key $(\mathfrak{a}, A)$ and $(\mathfrak{b}, B)$ then they can compute the shared key $E_{AB} = [\mathfrak{a}][\mathfrak{b}] \cdot E_0 = [\mathfrak{b}][\mathfrak{a}] \cdot E_0$.

MOTIVATING OSIDH

The constraint to $\mathbb{F}_p$-rational isogenies can be interpreted as an orientation of the supersingular graph by the subring $\mathbb{Z}[\pi]$ of $\mathbf{End}(E)$ generated by the Frobenius endomorphism π.

We introduce a general notion of orienting supersingular elliptic curves and their isogenies, and use this as the basis to construct a general oriented supersingular isogeny Diffie-Hellman (OSIDH) protocole.

Motivation

- Generalize CSIDH.
- Key space of SIDH: in order to have the two key spaces of similar size, we need to take $\ell_A^{e_A} \approx \ell_B^{e_B} \approx \sqrt{p}$. This implies that the space of choices for the secret key is limited to a fraction of the whole set of supersingular j-invariants over $\mathbb{F}_{p^2}$.
- A feature shared by SIDH and CSIDH is that the isogenies are constructed as quotients of rational torsion subgroups. The need for rational points limits the choice of the prime p

ORIENTATIONS

Let $\mathcal{O}$ be an order in an imaginary quadratic field. An *$\mathcal{O}$-orientation* on a supersingular elliptic curve E is an inclusion $\iota : \mathcal{O} \hookrightarrow \mathsf{End}(E)$, and a *$K$-orientation* is an inclusion $\iota : K \hookrightarrow \mathsf{End}^0(E) = \mathsf{End}(E) \otimes_{\mathbb{Z}} \mathbb{Q}$. An $\mathcal{O}$-orientation is *primitive* if $\mathcal{O} \simeq \mathsf{End}(E) \cap \iota(K)$.

Theorem

The category of K-oriented supersingular elliptic curves (E, ι), whose morphisms are isogenies commuting with the K-orientations, is equivalent to the category of elliptic curves with CM by K.

Let $\phi : E \to F$ be an isogeny of degree ℓ. A K-orientation $\iota : K \hookrightarrow \mathsf{End}^0(E)$ determines a K-orientation $\phi_*(\iota) : K \hookrightarrow \mathsf{End}^0(F)$ on F, defined by

$$\phi_*(\iota)(\alpha) = \frac{1}{\ell}\,\phi \circ \iota(\alpha) \circ \hat{\phi}.$$

Conversely, given K-oriented elliptic curves (E, ι_E) and (F, ι_F) we say that an isogeny $\phi : E \to F$ is K-oriented if $\phi_*(\iota_E) = \iota_F$, i.e., if the orientation on F is induced by ϕ.

ORIENTED ELLIPTIC CURVES AND VOLCANOES

As we have seen, one feature of the ℓ-isogeny graphs of CM elliptic curves is that in each component, depending on whether ℓ is split, inert, or ramified in K, there is a cycle of vertices, unique vertex, or adjacent pair of vertices which have ℓ-maximal endomorphism ring.

Chains of ℓ-isogenies leading away from these ℓ-maximal vertices have successively (and strictly) smaller endomorphism rings, by a power of ℓ.

This lets us define the depth of a CM elliptic curve E (i.e. vertex) in the ℓ-isogeny graph as the valuation of the index $[\mathcal{O}_K : \mathrm{End}(E)]$ at ℓ, which measures the distance to an ℓ-maximal vertex.

Consequently, we obtain a notion of depth at ℓ in the K-oriented supersingular ℓ-isogeny graph.

We also recover the notion of horizontal, ascending and descending isogenies.

CLASS GROUP ACTION

- $\mathsf{SS}(p) = \{$supersingular elliptic curves over $\overline{\mathbb{F}}_p$ up to isomorphism$\}$.
- $\mathsf{SS}_{\mathcal{O}}(p) = \{\mathcal{O}$-oriented s.s. elliptic curves over $\overline{\mathbb{F}}_p$ up to K-isomorphism$\}$.
- $\mathsf{SS}_{\mathcal{O}}^{pr}(p) =$subset of primitive $\mathcal{O}$-oriented curves.

The set $\mathsf{SS}_{\mathcal{O}}(p)$ admits a transitive group action:

$$\mathcal{C}\ell(\mathcal{O}) \times \mathsf{SS}_{\mathcal{O}}(p) \longrightarrow \mathsf{SS}_{\mathcal{O}}(p) \qquad ([\mathfrak{a}], E) \longmapsto [\mathfrak{a}] \cdot E = E/E[\mathfrak{a}]$$

Proposition

The class group $\mathcal{C}\ell(\mathcal{O})$ acts faithfully and transitively on the set of $\mathcal{O}$-isomorphism classes of primitive $\mathcal{O}$-oriented elliptic curves.

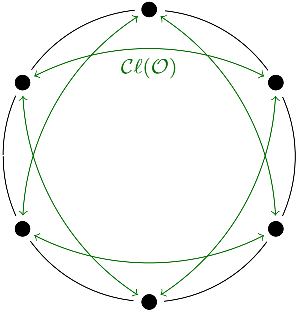
In particular, for fixed primitive $\mathcal{O}$-oriented E, we obtain a bijection of sets:

$$\mathcal{C}\ell(\mathcal{O}) \longrightarrow \mathsf{SS}_{\mathcal{O}}^{pr}(p) \qquad [\mathfrak{a}] \longmapsto [\mathfrak{a}] \cdot E$$

For any ideal class $[\mathfrak{a}]$ and generating set $\{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$ of small primes, coprime to $[\mathcal{O}_K : \mathcal{O}]$, we can find an identity $[\mathfrak{a}] = [\mathfrak{q}_1^{e_1} \cdot \ldots \cdot \mathfrak{q}_r^{e_r}]$, in order to compute the action via a sequence of low-degree isogenies.

Vortex

We define a vortex to be the ℓ-isogeny subgraph whose vertices are isomorphism classes of $\mathcal{O}$-oriented elliptic curves with ℓ-maximal endomorphism ring, equipped with an action of $\mathcal{C}\ell(\mathcal{O})$.

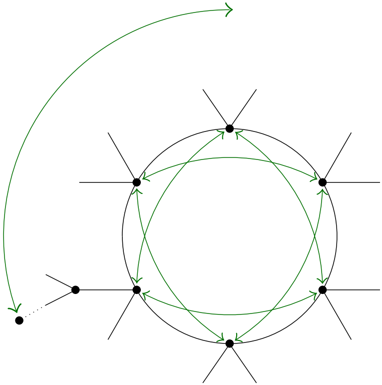


Instead of considering the union of different isogeny graphs, we focus on one single crater and we think of all the other primes as acting on it: the resulting object is a single isogeny circle rotating under the action of $\mathcal{C}\ell(\mathcal{O})$.

WHIRLPOOL

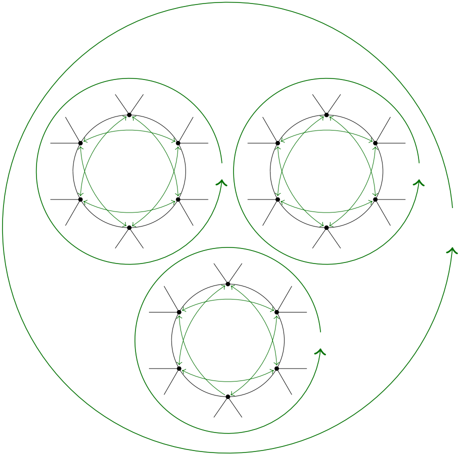
The action of $\mathcal{C}\ell(\mathcal{O})$ extends to the union $\bigcup_i SS_{\mathcal{O}_i}(p)$ over all superorders $\mathcal{O}_i$ containing $\mathcal{O}$ via the surjections $\mathcal{C}\ell(\mathcal{O}) \to \mathcal{C}\ell(\mathcal{O}_i)$.

We define a *whirlpool* to be a complete isogeny volcano acted on by the class group. We would like to think at isogeny graphs as moving objects.



Whirlpool

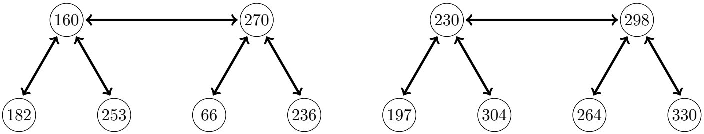
Actually, we would like to take the ℓ-isogeny graph on the full $\mathcal{C}\ell(\mathcal{O}_K)$-orbit. This might be composed of several ℓ-isogeny orbits (craters), although the class group is transitive.



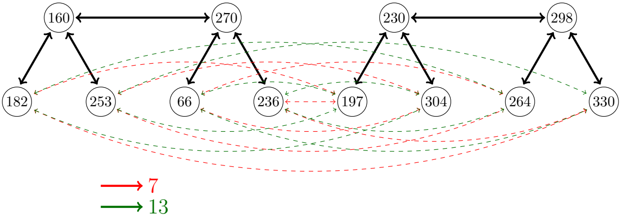
WHIRLPOOL: AN EXAMPLE

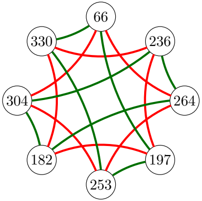
The set of multiple ℓ-volcanoes is called ℓ-cordillera.

Example. $p = 353, \ell = 2$, elliptic curves with 344 $\mathbb{F}_{353}$-rational points.



A whirlpool is the union of the two, shuffled by the class group of $\mathbb{Z}[2\sqrt{-82}]$.





Isogeny chains

Definition

An ℓ-isogeny chain of length n from E_0 to E is a sequence of isogenies of degree ℓ:

$$E_0 \xrightarrow{\phi_0} E_1 \xrightarrow{\phi_1} E_2 \xrightarrow{\phi_2} \dots \xrightarrow{\phi_{n-1}} E_n = E.$$

The ℓ-isogeny chain is without backtracking if $\mathtt{ker}\,(\phi_{i+1} \circ \phi_i) \neq E_i[\ell],\ \forall i$. The isogeny chain is descending (or ascending, or horizontal) if each ϕ_i is descending (or ascending, or horizontal, respectively).

The dual isogeny of ϕ_i is the only isogeny ϕ_{i+1} satisfying $\mathtt{ker}\,(\phi_{i+1} \circ \phi_i) = E_i[\ell]$. Thus, an isogeny chain is without backtracking if and only if the composition of two consecutive isogenies is cyclic.

Lemma

The composition of the isogenies in an ℓ-isogeny chain is cyclic if and only if the ℓ-isogeny chain is without backtracking.

PUSHING ISOGENIES ALONG A CHAIN

Suppose that (E_i, ϕ_i) is an ℓ-isogeny chain, with E_0 equipped with an $\mathcal{O}_K$-orientation $\iota_0 : \mathcal{O}_K \to \mathrm{End}(E_0)$.

For each i, $\iota_i : K \to \mathrm{End}^0(E_i)$ is the induced K-orientation on E_i. Write $\mathcal{O}_i = \mathrm{End}(E_i) \cap \iota_i(K)$ with $\mathcal{O}_0 = \mathcal{O}_K$.

If $\mathfrak{q}$ is a split prime in $\mathcal{O}_K$ over $q \neq \ell, p$, then the isogeny

$$\psi_0 : E_0 \to F_0 = E_0/E_0\left[\mathfrak{q}\right]$$

can be extended to the ℓ-isogeny chain by pushing forward $C_0 = E_0\left[\mathfrak{q}\right]$:

$$C_0 = E_0\left[\mathfrak{q}\right],\ C_1 = \phi_0(C_0), \dots,\ C_n = \phi_{n-1}(C_{n-1})$$

and defining $F_i = E_i/C_i$.

$$\begin{tikzcd}
E_{i-1}/C_{i-1} = F_{i-1} \arrow[r, "\ell"] & F_i = E_i/C_i \\
C_{i-1} \subseteq E_{i-1} \arrow[u, "\psi_{i-1}", "\mathfrak{q}"'] \arrow[r, "\phi_{i-1}", "\ell"'] & E_i \supseteq C_i \arrow[u, "\psi_i", "\mathfrak{q}"']
\end{tikzcd}$$

LADDERS

Definition

An ℓ-ladder of length n and degree q is a commutative diagram of ℓ-isogeny chains (E_i, ϕ_i), (F_i, ϕ_i') of length n connected by q-isogenies $\psi_i : E_i \to F_i$

$$\begin{tikzcd} E_0 \arrow[r, "\phi_0"] \arrow[d, "\psi_0"'] & E_1 \arrow[r, "\phi_1"] \arrow[d, "\psi_1"'] & E_2 \arrow[r, "\phi_2"] \arrow[d, "\psi_2"'] & \cdots \arrow[r, "\phi_{n-1}"] & E_n \arrow[d, "\psi_n"'] \\ F_0 \arrow[r, "\phi_0'"'] & F_1 \arrow[r, "\phi_1'"'] & F_2 \arrow[r, "\phi_2'"'] & \cdots \arrow[r, "\phi_{n-1}'"'] & F_n \end{tikzcd}$$

We also refer to an ℓ-ladder of degree q as a q-isogeny of ℓ-isogeny chains.

We say that an ℓ-ladder is ascending (or descending, or horizontal) if the ℓ-isogeny chain (E_i, ϕ_i) is ascending (or descending, or horizontal, respectively).

We say that the ℓ-ladder is level if ψ_0 is a horizontal q-isogeny. If the ℓ-ladder is descending (or ascending), then we refer to the length of the ladder as its depth (or, respectively, as its height).

EFFECTIVE ENDOMORPHISM RINGS AND ISOGENIES

We say that a subring of $\text{End}(E)$ is effective if we have explicit polynomials or rational functions which represent its generators.

Examples. $\mathbb{Z}$ in $\text{End}(E)$ is effective. Effective imaginary quadratic subrings $\mathcal{O} \subset \text{End}(E)$, are the subrings $\mathcal{O} = \mathbb{Z}[\pi]$ generated by Frobenius

In the Couveignes-Rostovtsev-Stolbunov constructions, or in the CSIDH protocol, one works with $\mathcal{O} = \mathbb{Z}[\pi]$.

- For large finite fields, the class group of $\mathcal{O}$ is large and the primes $\mathfrak{q}$ in $\mathcal{O}$ have no small generators.
  Factoring the division polynomial $\psi_q(x)$ to find the kernel polynomial of degree $(q-1)/2$ for $E[\mathfrak{q}]$ becomes relatively expensive.
- In SIDH, the ordinary protocol of De Feo, Smith, and Kieffer, or CSIDH, the curves are chosen such that the points of $E[\mathfrak{q}]$ are defined over a small degree extension κ/k, and working with rational points in $E(\kappa)$.
- We propose the use of an effective CM order $\mathcal{O}_K$ of class number 1.
  The kernel polynomial can be computed directly without need for a splitting field for $E[\mathfrak{q}]$, and the computation of a generator isogeny is a one-time precomputation.

Modular approach

The use of modular curves for efficient computation of isogenies has an established history (see Elkies)

Modular Curve

The modular curve $\mathtt{X}(1) \simeq \mathbb{P}^1$ classifies elliptic curves up to isomorphism, and the function j generates its function field.

The modular polynomial $\Phi_m(X, Y)$ defines a correspondence in $\mathtt{X}(1) \times \mathtt{X}(1)$ such that $\Phi_m(j(E), j(E')) = 0$ if and only if there exists a cyclic m-isogeny ϕ from E to E', possibly over some extension field.

Definition

A *modular ℓ-isogeny chain* of length n over k is a finite sequence $(j_0, j_1, \dots, j_n)$ in k such that $\Phi_\ell(j_i, j_{i+1}) = 0$ for $0 \leq i < n$.
A *modular ℓ-ladder* of length n and degree q over k is a pair of modular ℓ-isogeny chains

$$(j_0, j_1, \dots, j_n) \text{ and } (j'_0, j'_1, \dots, j'_n),$$

such that $\Phi_q(j_i, j'_i) = 0$.

OSIDH - Introduction

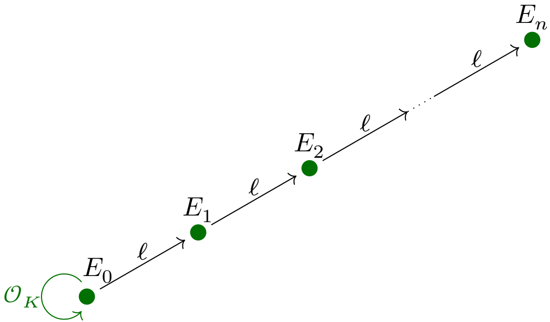
We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

$$E_0 \xrightarrow{\ell} E_1 \xrightarrow{\ell} E_2 \xrightarrow{\ell} \cdots \xrightarrow{\ell} E_n$$

OSIDH - Introduction

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

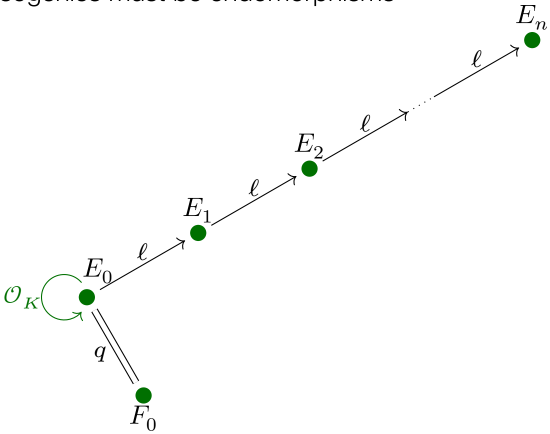
- For $\ell = 2$ (or 3) a suitable candidate for $\mathcal{O}_K$ could be the Gaussian integers $\mathbb{Z}[i]$ or the Eisenstein integers $\mathbb{Z}[\omega]$.



OSIDH - Introduction

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

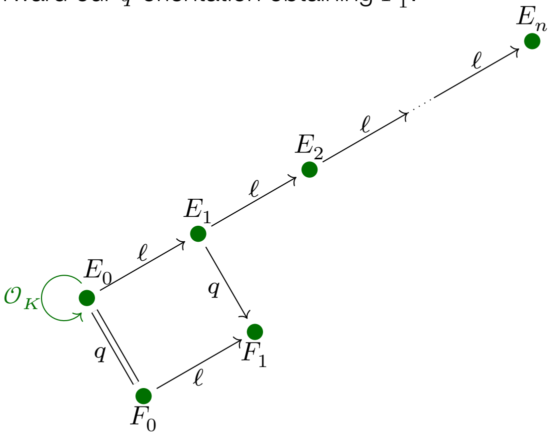
- Horizontal isogenies must be endomorphisms



OSIDH - Introduction

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

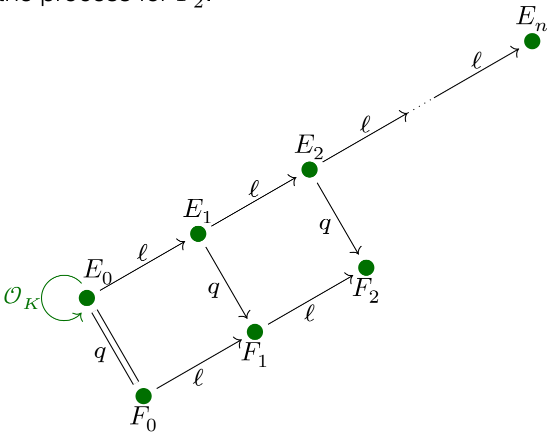
- We push forward our q-orientation obtaining F_1.



OSIDH - Introduction

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

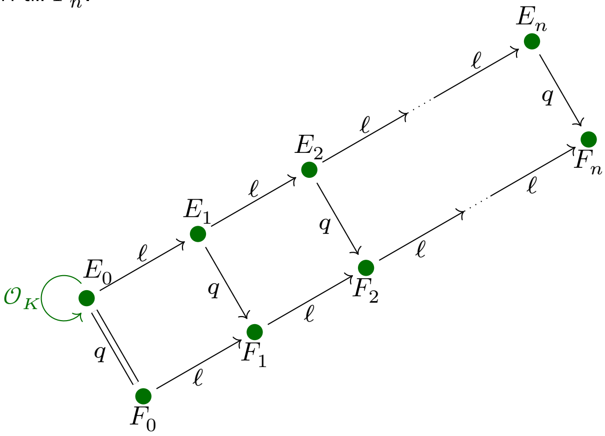
- We repeat the process for F_2.



OSIDH - Introduction

We consider an elliptic curve E_0 with an effective endomorphism ring (eg. $j_0 = 0, 1728$) and a chain of ℓ-isogenies.

- And again till F_n.



HOW FAR SHOULD WE GO?

In order to have the action of $\mathcal{C}\ell(\mathcal{O})$ cover a large portion of the supersingular elliptic curves, we require $\ell^n \sim p$, i.e., $n \sim \mathtt{log}_\ell(p)$.

- $\#SS^{pr}_{\mathcal{O}}(p) = h(\mathcal{O}_n)$ =class number of $\mathcal{O}_n = \mathbb{Z} + \ell^n \mathcal{O}_K$.
- Class Number Formula

$$h(\mathbb{Z} + m\mathcal{O}_K) = \frac{h(\mathcal{O}_K)m}{[\mathcal{O}_K^\times : \mathcal{O}^\times]} \prod_{p|m} \left(1 - \left(\frac{\Delta_K}{p}\right)\frac{1}{p}\right)$$

- Units

$$\mathcal{O}_K^\times = \begin{cases} \{\pm 1\} & \text{if } \Delta_K < -4 \\ \{\pm 1, \pm i\} & \text{if } \Delta_K = -4 \\ \{\pm 1, \pm\omega, \pm\omega^2\} & \text{if } \Delta_K = -3 \end{cases} \Rightarrow [\mathcal{O}_K^\times : \mathcal{O}^\times] = \begin{cases} 1 & \text{if } \Delta_K < -4 \\ 2 & \text{if } \Delta_K = -4 \\ 3 & \text{if } \Delta_K = -3 \end{cases}$$

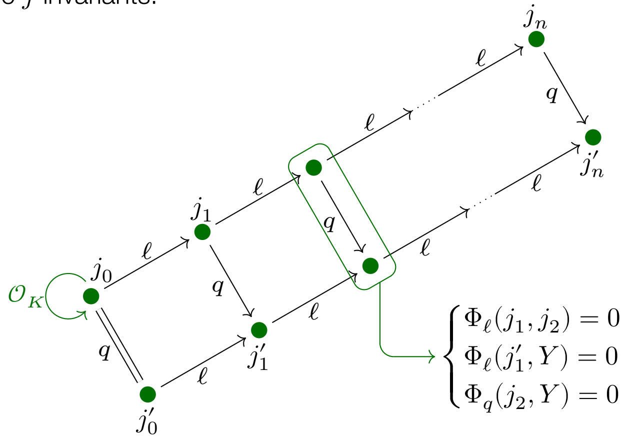
- Number of Supersingular curves

$$\#\mathrm{SS}(p) = \left[\frac{p}{12}\right] + \epsilon_p \qquad \epsilon_p \in \{0, 1, 2\}$$

Therefore, $h(\ell^n \mathcal{O}_K) = \dfrac{1 \cdot \ell^n}{2 \text{ or } 3}\left(1 - \left(\dfrac{\Delta_K}{\ell}\right)\dfrac{1}{\ell}\right) = \left[\dfrac{p}{12}\right] + \epsilon_p \implies p \sim \ell^n$

OSIDH - Introduction & Modular Approach

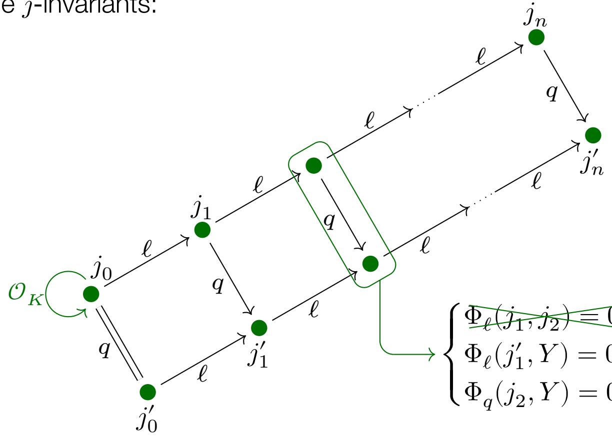
If we look at modular polynomials $\Phi_\ell(X,Y)$ and $\Phi_q(X,Y)$ we realize that all we need are the j-invariants:



$$\begin{cases} \Phi_\ell(j_1, j_2) = 0 \\ \Phi_\ell(j_1', Y) = 0 \\ \Phi_q(j_2, Y) = 0 \end{cases}$$

OSIDH - Introduction & Modular Approach

If we look at modular polynomials $\Phi_\ell(X,Y)$ and $\Phi_q(X,Y)$ we realize that all we need are the j-invariants:



$$\begin{cases} \cancel{\Phi_\ell(j_1, j_2) = 0} \\ \Phi_\ell(j'_1, Y) = 0 \\ \Phi_q(j_2, Y) = 0 \end{cases}$$

Since j_2 is given (the initial chain is known) and supposing that j'_1 has already been constructed, j'_2 is determined by a system of two equations

HOW MANY STEPS BEFORE THE IDEALS ACT DIFFERENTLY?

$$
\begin{tikzcd}
E_0 \arrow[r, shift left, "\mathfrak{q}"] \arrow[d] \arrow[rr, bend left, "\mathfrak{q}^2"] & E_0 \arrow[l, shift left, "\bar{\mathfrak{q}}"] \arrow[r, "\mathfrak{q}"] \arrow[d] & E_0 \arrow[d] \\
E_1'' \arrow[d] & E_1 \arrow[l] \arrow[r] \arrow[d] & E_1' \arrow[d] \\
{} & {} & {}
\end{tikzcd}
$$

$E_i' \neq E_i''$ if and only if $\mathfrak{q}^2 \cap \mathcal{O}_i$ is not principal and the probability that a random ideal in $\mathcal{O}_i$ is principal is $1/h(\mathcal{O}_i)$. In fact, we can do better; we write $\mathcal{O}_K = \mathbb{Z}[\omega]$ and we observe that if $\mathfrak{q}^2$ was principal, then

$$q^2 = \mathsf{N}(\mathfrak{q}^2) = \mathsf{N}(a + b\ell^i\omega)$$

since it would be generated by an element of $\mathcal{O}_i = \mathbb{Z} + \ell^i\mathcal{O}_K$. Now

$$\mathsf{N}(a + b\ell^i) = a^2 \pm abt\ell^i + b^2 s\ell^{2i} \quad \text{where} \quad \omega^2 + t\omega + s = 0$$

Thus, as soon as $\ell^{2i} \gg q^2$, we are guaranteed that $\mathfrak{q}^2$ is not principal.

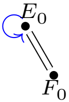
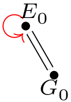
A FIRST NAIVE PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to ... \to E_n$

ALICE	**BOB**

A FIRST NAIVE PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to ... \to E_n$

	ALICE	**BOB**
Choose a primitive $\mathcal{O}_K$-orientation of E_0		

A FIRST NAIVE PROTOCOL

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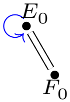
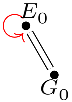
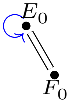
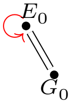
Choose a primitive $\mathcal{O}_K$-orientation of E_0		
Push it forward to depth n	$\underbrace{E_0 = F_0 \to F_1 \to \ldots \to F_n}_{\phi_A}$	$\underbrace{E_0 = G_0 \to G_1 \to \ldots \to G_n}_{\phi_B}$

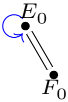
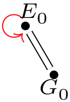
A FIRST NAIVE PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \ldots \to E_n$

	ALICE	**BOB**
Choose a primitive $\mathcal{O}_K$-orientation of E_0		
Push it forward to depth n	$\underbrace{E_0 = F_0 \to F_1 \to \ldots \to F_n}_{\phi_A}$	$\underbrace{E_0 = G_0 \to G_1 \to \ldots \to G_n}_{\phi_B}$
Exchange data	$\{G_i\}_{i=1}^n$	$\{F_i\}_{i=1}^n$

A FIRST NAIVE PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \dots \to E_n$

	ALICE	**BOB**
Choose a primitive $\mathcal{O}_K$-orientation of E_0		
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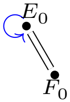
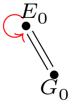
Exchange data	$\{G_i\}_{i=1}^n$	$\{F_i\}_{i=1}^n$
Compute shared secret	Compute $\phi_A \cdot \{G_i\}$	Compute $\phi_B \cdot \{F_i\}$

A FIRST NAIVE PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \ldots \to E_n$

	ALICE	**BOB**
Choose a primitive $\mathcal{O}_K$-orientation of E_0		
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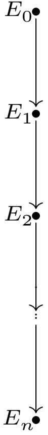
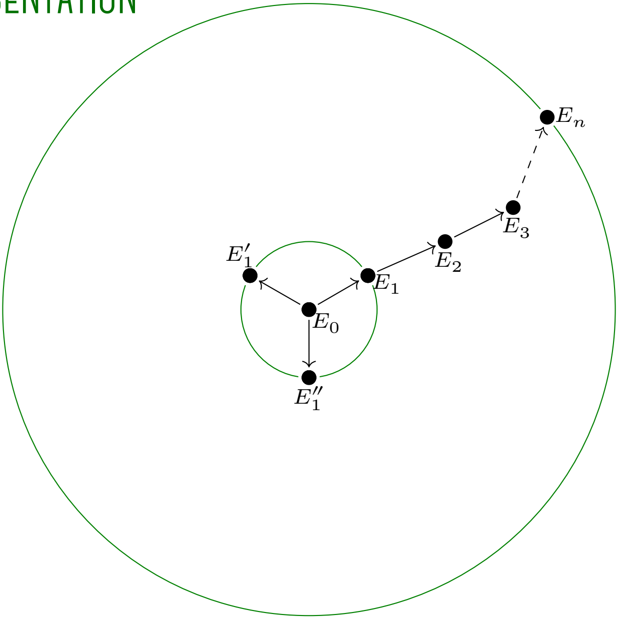
Exchange data	$\{G_i\}_{i=1}^{n}$	$\{F_i\}_{i=1}^{n}$
Compute shared secret	Compute $\phi_A \cdot \{G_i\}$	Compute $\phi_B \cdot \{F_i\}$

In the end, Alice and Bob will share a new chain $E_0 \to H_1 \to \ldots \to H_n$

GRAPHIC REPRESENTATION





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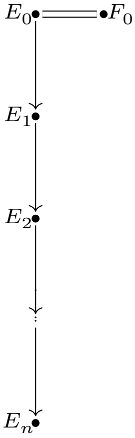
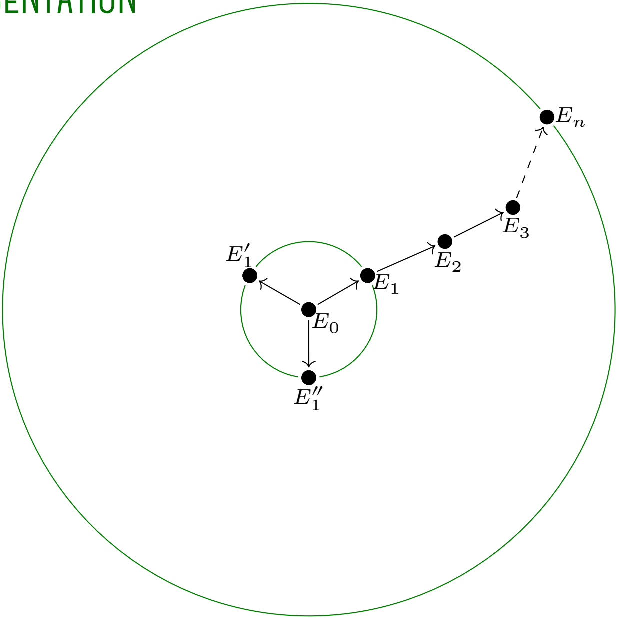
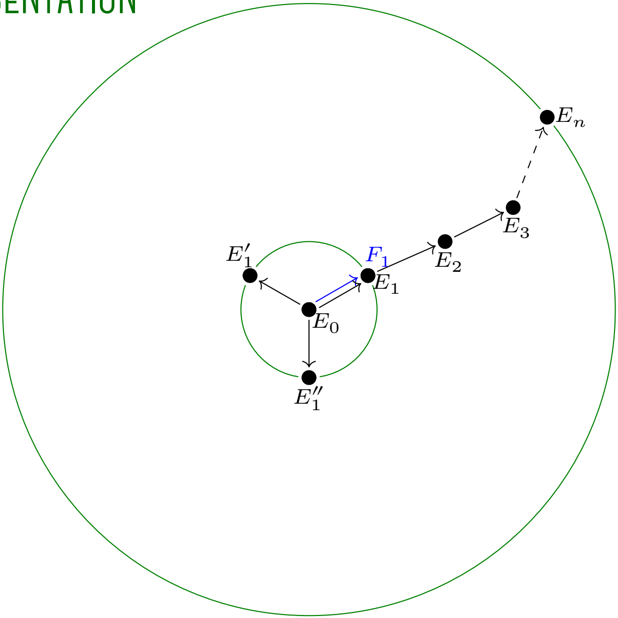




GRAPHIC REPRESENTATION

$$\begin{tikzcd}
E_0 \arrow[r, equal] \arrow[d] & F_0 \arrow[d] \\
E_1 \arrow[r] \arrow[d] & F_1 \\
E_2 \arrow[d] & \\
\vdots \arrow[d] & \\
E_n &
\end{tikzcd}$$



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$$\begin{tikzcd}
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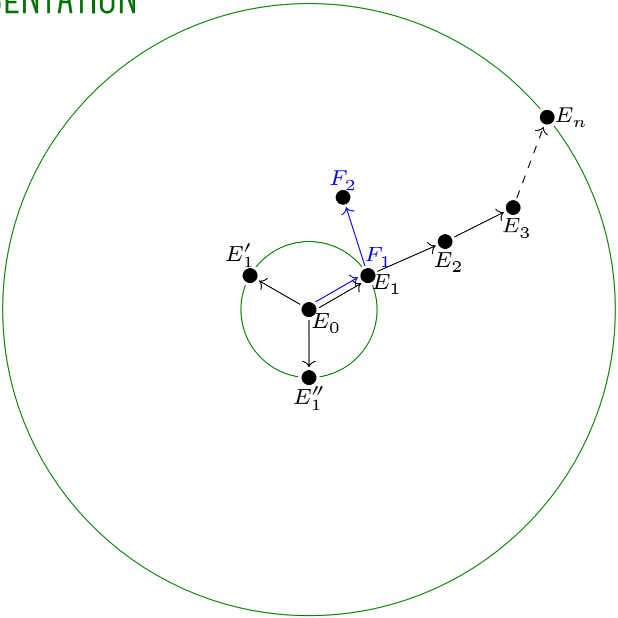
E_1 \arrow[r] \arrow[d] & F_1 \arrow[d] \\
E_2 \arrow[r] \arrow[d] & F_2 \\
\vdots \arrow[d] & \\
E_n &
\end{tikzcd}$$



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$$\begin{tikzcd}
E_0 \arrow[r, equal] \arrow[d] & F_0 \arrow[d] \\
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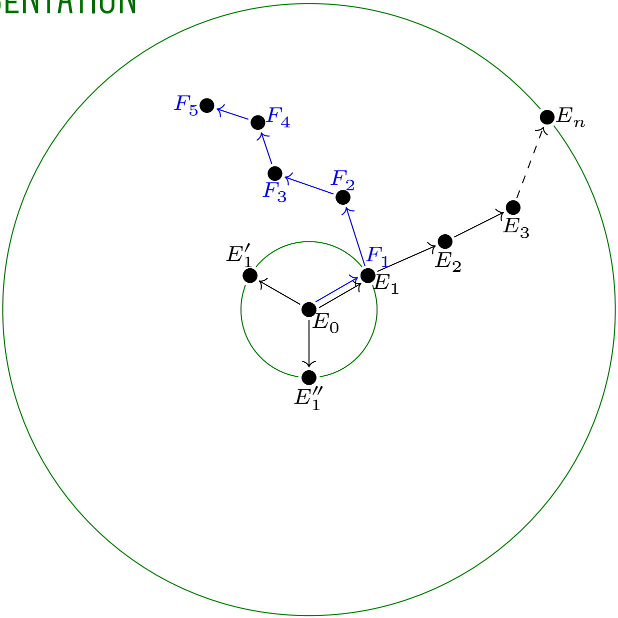
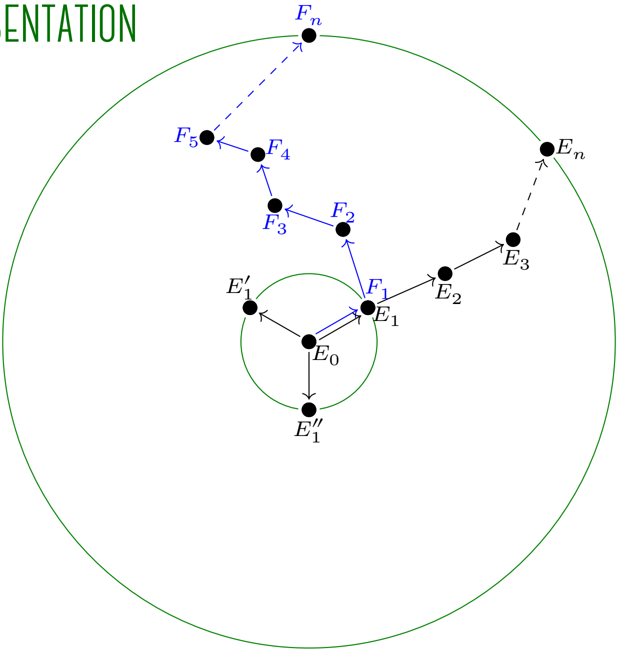
\vdots \arrow[d] & \vdots \\
E_n &
\end{tikzcd}$$



GRAPHIC REPRESENTATION

$$\begin{tikzcd}
E_0 \arrow[r, equal] \arrow[d] & F_0 \arrow[d] \\
E_1 \arrow[r] \arrow[d] & F_1 \arrow[d] \\
E_2 \arrow[r] \arrow[d] & F_2 \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
E_n \arrow[r] & F_n
\end{tikzcd}$$



GRAPHIC REPRESENTATION

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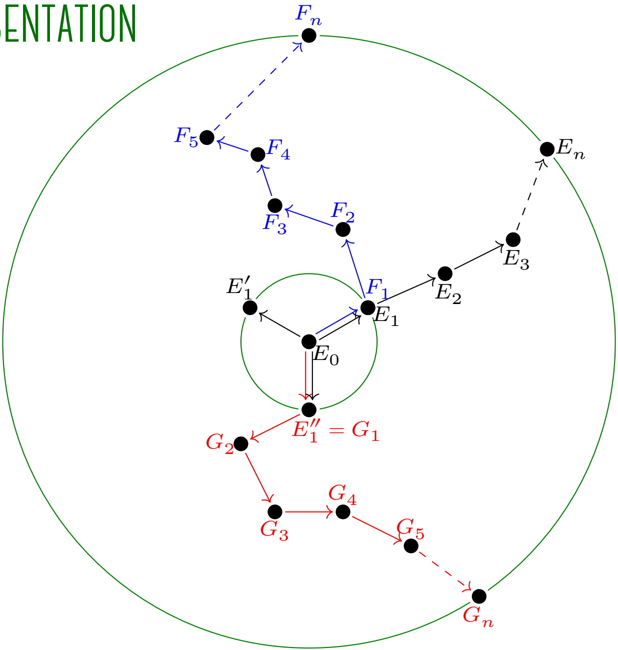
E_0 \arrow[r, equal] \arrow[d] & G_0 \arrow[d] \\
E_1 \arrow[r] \arrow[d] & G_1 \arrow[d] \\
E_2 \arrow[r] \arrow[d] & G_2 \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
E_n \arrow[r] & G_n
\end{tikzcd}$$



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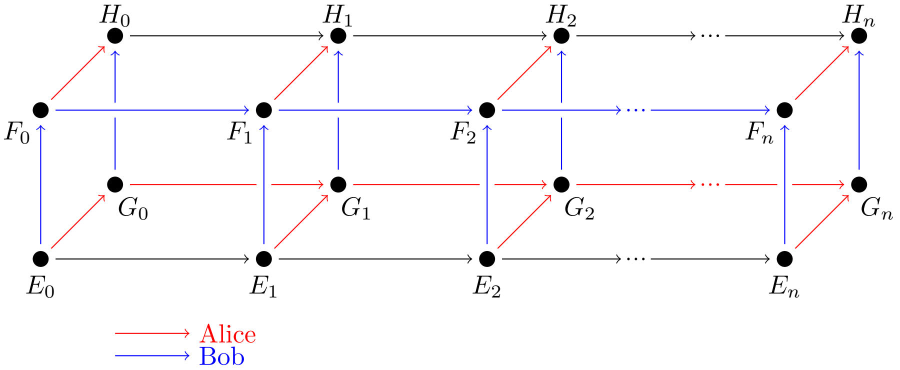


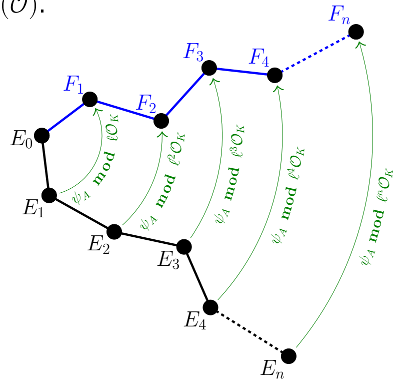
A FIRST NAIVE PROTOCOL - WEAKNESS

In reality, sharing (F_i) and (G_i) reveals too much of the private data.

From the short exact sequence of class groups:

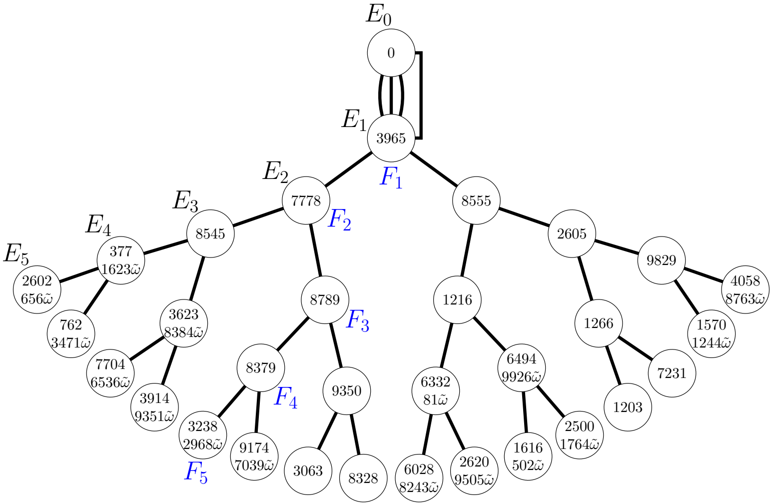
$$1 \to \frac{(\mathcal{O}_K/\ell^n\mathcal{O}_K)^\times}{\mathcal{O}_K^\times(\mathbb{Z}/\ell^n\mathbb{Z})^\times} \to \mathcal{C}\ell(\mathcal{O}) \to \mathcal{C}\ell(\mathcal{O}_K) \to 1$$

an adversary can compute successive approximations (mod ℓ^i) to ϕ_A and ϕ_B modulo ℓ^n hence in $\mathcal{C}\ell(\mathcal{O})$.



AN EXAMPLE: COMPUTE SUCCESSIVE APPROXIMATIONS

Take $q = p^2 = 10007^2$. $E_0 : y^2 = x^3 + 1$ of j-invariant 0 is supersingular over $\mathbb{F}_q$.
We orient E_0 by $\mathcal{O}_K = \mathbb{Z}[\omega] \hookrightarrow \mathrm{End}(E_0)$ where $w^2 + w + 1$.



AN EXAMPLE: COMPUTE SUCCESSIVE APPROXIMATIONS

Algorithm. Action of an ideal $\left[(q, a+b\ell^i w)\right] \in \mathcal{C}\ell(\mathbb{Z}+\ell^i\mathcal{O}_K)$ lying over q on the set of primitive $\mathcal{O}$-oriented elliptic curves $\mathrm{SS}_{\mathcal{O}}^{pr}(p)$.

Input: The j-invariants of two elliptic curves E and E' over $\mathbb{F}_{p^2}$ known to be q-isogenous.
Output: The ideal $[\mathfrak{a}] \in \{[\mathfrak{q}], [\overline{\mathfrak{q}}]\}$ such that $[\mathfrak{a}] * j(E) = j(E')$.

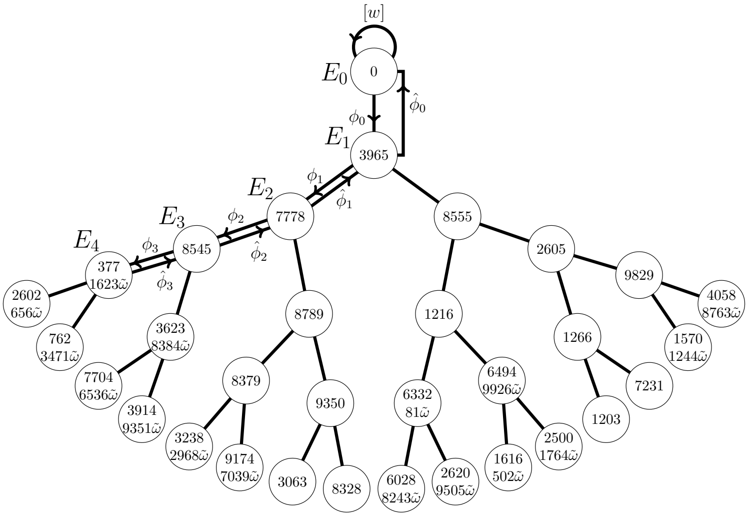
---

1. Compute q-division polynomial $\psi_q(x)$.
2. Factor $\psi_q(x)$ and find the factor $f(x)$ corresponding to the desired isogeny $\phi : E \to E'$.
3. Pick a root of f, i.e., a q-torsion point P lying in the kernel of ϕ.
4. Set $m\mathcal{O} = \mathfrak{q}\overline{\mathfrak{q}} = (q, a+b\ell^i w)(q, a'+b'\ell^i w)$.
5. **If** $[a]\,P + [b]\cdot\left[\ell^i w\right] P = O_E$
   Return $\mathfrak{q}$.
   Else
   Return $\overline{\mathfrak{q}}$.

AN EXAMPLE: COMPUTE SUCCESSIVE APPROXIMATIONS

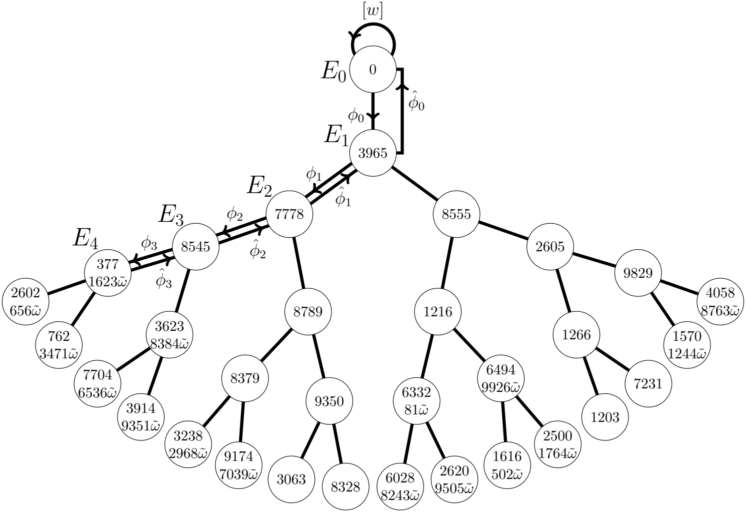
The action of $\ell^i\omega$ on E_i will be given by the composition

$$\phi_{i-1} \circ \cdots \circ \phi_2 \circ \phi_1 \circ \phi_0 \circ [\omega] \circ \hat{\phi}_0 \circ \hat{\phi}_1 \circ \hat{\phi}_2 \circ \cdots \circ \hat{\phi}_{i-1}$$



AN EXAMPLE: COMPUTE SUCCESSIVE APPROXIMATIONS

Observe that this is exactly the definition of orientation by $\mathcal{O}_i$ transmitted to E_i along the isogeny $E_0 \to E_1 \to E_2 \to \ldots \to E_i$.



THE ALGORITHM

Computing successive approximations

We are given two sequences $\{E_i\}_{i=0}^n$ and $\{F_i\}_{i=0}^n$. Suppose that $E_i = F_i$ for all $i \leq m$; there are l possibilities for F_{m+1}, and we need to find $\beta \in \mathrm{End}(\mathcal{O}_K)$ such that

1. $\beta \equiv 1 \bmod \ell^m$ so that $\beta_* E_i = F_i = E_i$ for all $i \leq m$;
2. $\beta_* E_{m+1} = F_{m+1}$;
3. β is smooth with small exponents (n order to determine the action of β modulo ℓ^{m+1} effectively).

Once that we have constructed α such that $\alpha_* E_i = F_i$ for all $m < i \leq k$, then we can substitute **1** with

1'. $\beta \equiv \alpha \bmod \ell^k$ so that $\beta_* E_{k+1} = F_{k+1}$.

TOWARDS A MORE SECURE OSIDH PROTOCOL

How can we avoid this while still giving the other enough information?

Instead Alice and Bob can send only $F = F_n$ and $G = G_n$.

Problem Once Alice receives the unoriented curve G_n computed by Bob she also needs additional information for each prime $\mathfrak{p}_i$:

Bob's curve

$$\begin{tikzcd} {} & G_n \arrow[l, "\text{Horizontal } p_i\text{-isogeny with kernel } G_n[\bar{\mathfrak{p}}_i]"'] \arrow[r, "\text{Horizontal } p_i\text{-isogeny with kernel } G_n[\mathfrak{p}_i]"'] & {} \end{tikzcd}$$

In fact, she has no information as to which directions — out of $p_i + 1$ total p_i-isogenies — to take as $\mathfrak{p}_i$ and $\bar{\mathfrak{p}}_i$.

Solution They share a collection of local isogeny data $(F_n[\mathfrak{q}_j])$ and $(G_n[\mathfrak{q}_j])$ which identifies the isogeny directions (out of $q_i + 1$) for a system of small split primes $(\mathfrak{q}_i)$ in $\mathcal{O}_K$.

OSIDH PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \ldots \to E_n$ and a set of splitting primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t \subseteq \mathcal{O} \subseteq \mathrm{End} E_n \cap K \subseteq \mathcal{O}_K$

ALICE	**BOB**

OSIDH PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \dots \to E_n$ and a set of splitting primes $\mathfrak{p}_1, \dots, \mathfrak{p}_t \subseteq \mathcal{O} \subseteq \mathrm{End} E_n \cap K \subseteq \mathcal{O}_K$

	ALICE	**BOB**
Choose integers in a bound $[-r, r]$	$(e_1, \dots, e_t)$	$(d_1, \dots, d_t)$

OSIDH PROTOCOL

PUBLIC DATA: A chain of ℓ-isogenies $E_0 \to E_1 \to \dots \to E_n$ and a set of splitting primes $\mathfrak{p}_1, \dots, \mathfrak{p}_t \subseteq \mathcal{O} \subseteq \mathrm{End} E_n \cap K \subseteq \mathcal{O}_K$

	ALICE	**BOB**
Choose integers in a bound $[-r, r]$	$(e_1, \dots, e_t)$	$(d_1, \dots, d_t)$
Construct an isogenous curve	$F_n = E_n / E_n \left[\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}\right]$	$G_n = E_n / E_n \left[\mathfrak{p}_1^{d_1} \cdots \mathfrak{p}_t^{d_t}\right]$

OSIDH PROTOCOL

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Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \ldots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \ldots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$

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Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$
... and their conjugates	$F_n \to F_{n,i}^{(1)} \to \dots \to F_{n,i}^{(r-1)} \to F_{n,1}^{(r)}$	$G_n \to G_{n,i}^{(1)} \to \dots \to G_{n,i}^{(r-1)} \to G_{n,1}^{(r)}$

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Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$
... and their conjugates	$F_n \to F_{n,i}^{(1)} \to \dots \to F_{n,i}^{(r-1)} \to F_{n,1}^{(r)}$	$G_n \to G_{n,i}^{(1)} \to \dots \to G_{n,i}^{(r-1)} \to G_{n,1}^{(r)}$
Exchange data	G_n+directions	F_n+directions

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Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$
... and their conjugates	$F_n \to F_{n,i}^{(1)} \to \dots \to F_{n,i}^{(r-1)} \to F_{n,1}^{(r)}$	$G_n \to G_{n,i}^{(1)} \to \dots \to G_{n,i}^{(r-1)} \to G_{n,1}^{(r)}$
Exchange data	G_n+directions	F_n+directions
Compute shared data	Takes e_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.	Takes d_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.

OSIDH PROTOCOL

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Construct an isogenous curve	$F_n = E_n / E_n \left[\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}\right]$	$G_n = E_n / E_n \left[\mathfrak{p}_1^{d_1} \cdots \mathfrak{p}_t^{d_t}\right]$
Precompute all directions $\forall i$	$F_{n,i}^{(-r)} \leftarrow F_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow F_{n,i}^{(1)} \leftarrow F_n$	$G_{n,i}^{(-r)} \leftarrow G_{n,i}^{(-r+1)} \leftarrow \dots \leftarrow G_{n,i}^{(1)} \leftarrow G_n$
... and their conjugates	$F_n \to F_{n,i}^{(1)} \to \dots \to F_{n,i}^{(r-1)} \to F_{n,1}^{(r)}$	$G_n \to G_{n,i}^{(1)} \to \dots \to G_{n,i}^{(r-1)} \to G_{n,1}^{(r)}$
Exchange data	G_n+directions	F_n+directions
Compute shared data	Takes e_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.	Takes d_i steps in $\mathfrak{p}_i$-isogeny chain & push forward information for $j > i$.

In the end, they share $H_n = E_n / E_n \left[\mathfrak{p}_1^{e_1+d_1} \cdot \ldots \cdot \mathfrak{p}_t^{e_t+d_t}\right]$

OSIDH PROTOCOL - GRAPHIC REPRESENTATION I

The first step consists of choosing the secret keys; these are represented by a sequence of integers $(e_1, \dots, e_t)$ such that $|e_i| \leq r$. The bound r is taken so that the number $(2r+1)^t$ of curves that can be reached is sufficiently large. This choice of integers enables Alice to compute a new elliptic curve

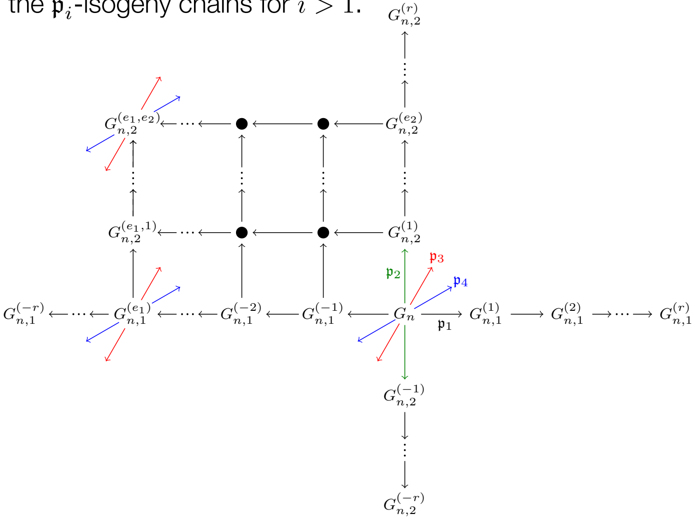
$$F_n = \frac{E_n}{E_n\left[\mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}\right]}$$

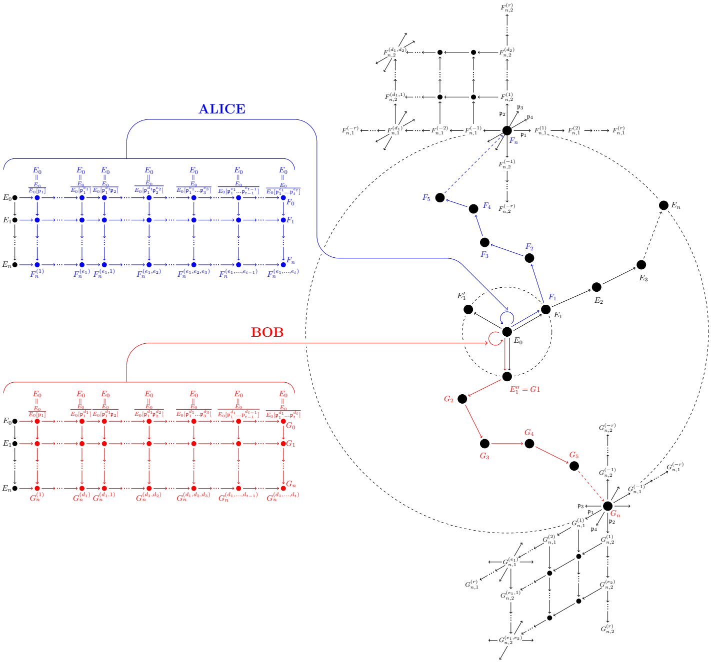
by means of constructing the following commutative diagram

$$\begin{tikzcd}
 & E_0 \arrow[d, equal] & & E_0 \arrow[d, equal] & E_0 \arrow[d, equal] & & E_0 \arrow[d, equal] & & E_0 \arrow[d, equal] & & E_0 \arrow[d, equal] \\
E_0 \arrow[r] \arrow[d] & \frac{E_0}{E_0[\mathfrak{p}_1]} \arrow[r] \arrow[d] & \cdots \arrow[r] & \frac{E_0}{E_0\left[\mathfrak{p}_1^{e_1}\right]} \arrow[r] \arrow[d] & \frac{E_0}{E_0\left[\mathfrak{p}_1^{e_1}\mathfrak{p}_2\right]} \arrow[r] \arrow[d] & \cdots \arrow[r] & \frac{E_0}{E_0\left[\mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\right]} \arrow[r] \arrow[d] & \cdots \arrow[r] & \frac{E_0}{E_0\left[\mathfrak{p}_1^{e_1}\dots\mathfrak{p}_{t-1}^{e_{t-1}}\right]} \arrow[r] \arrow[d] & \cdots \arrow[r] & \frac{E_0}{E_0\left[\mathfrak{p}_1^{e_1}\dots\mathfrak{p}_t^{e_t}\right]} = F_0 \arrow[d] \\
E_1 \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & \bullet \arrow[r] \arrow[d] & \cdots \arrow[r] & F_1 \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] & & \vdots \arrow[d] \\
E_n \arrow[r] & F_n^{(1)} \arrow[r] & \cdots \arrow[r] & F_n^{(e_1)} \arrow[r] & F_n^{(e_1,1)} \arrow[r] & \cdots \arrow[r] & F_n^{(e_1,e_2)} \arrow[r] & \cdots \arrow[r] & F_n^{(e_1,\dots,e_{t-1})} \arrow[r] & \cdots \arrow[r] & F_n = F_n^{(e_1,\dots,e_t)}
\end{tikzcd}$$

OSIDH PROTOCOL - GRAPHIC REPRESENTATION II

Once that Alice obtain from Bob the curve G_n together with the collection of data encoding the directions, she takes e_1 steps in the $\mathfrak{p}_1$-isogeny chain and push forward all the $\mathfrak{p}_i$-isogeny chains for $i > 1$.




ALICE
BOB
E_0
E_1
E_n
F_0
F_1
F_n
$F_n^{(1)}$
$F_n^{(e_1)}$
$F_n^{(e_1,1)}$
$F_n^{(e_1,e_2)}$
$F_n^{(e_1,e_2,e_3)}$
$F_n^{(e_1,\dots,e_{t-1})}$
$F_n^{(e_1,\dots,e_t)}$
G_0
G_1
G_n
$G_n^{(1)}$
$G_n^{(d_1)}$
$G_n^{(d_1,1)}$
$G_n^{(d_1,d_2)}$
$G_n^{(d_1,d_2,d_3)}$
$G_n^{(d_1,\dots,d_{t-1})}$
$G_n^{(d_1,\dots,d_t)}$
E_1'
$E_1'' = G1$
E_2
E_3
F_2
F_3
F_4
F_5
G_2
G_3
G_4
G_5

CLASSICAL HARD PROBLEMS

Endomorphism ring problem

Given a supersingular elliptic curve $E/\mathbb{F}_{p^2}$ and $\pi = [p]$, determine

1. End(E) as an abstract ring.
2. An explicit endomorphism $\phi \in$ End$(E) - \mathbb{Z}$.
3. An explicit basis $\mathfrak{B}^0$ for End$^0(E)$ over $\mathbb{Q}$.
4. An explicit basis $\mathfrak{B}$ for End(E) over $\mathbb{Z}$.

Endomorphism ring transfer problem

Given an isogeny chain

$$E_0 \longrightarrow E_1 \longrightarrow \ldots \longrightarrow E_n$$

and End(E_0), determine End(E_n).

HARD PROBLEMS

Endomorphism Generators Problem

Given a supersingular elliptic curve $E/\mathbb{F}_{p^2}$, $\pi = [p]$, an imaginary quadratic order $\mathcal{O}$ admitting an embedding in $\mathrm{End}(E)$ and a collection of compatible $(\mathcal{O}, \mathfrak{q}^n)$-orientations of E for $(\mathfrak{q}, n) \in S$, determine

1. An explicit endomorphism $\phi \in \mathcal{O} \subseteq \mathrm{End}(E)$
2. A generator ϕ of $\mathcal{O} \subseteq \mathrm{End}(E)$

Suppose $S = \{(\mathfrak{q}, n)\} = \{(\mathfrak{q}_1, n_1), \dots, (\mathfrak{q}_t, n_t)\}$ where $\mathfrak{q}_1, \dots, \mathfrak{q}_t$ are pairwise distinct primes such that

$$\begin{aligned}[0, \dots, n_1] \times \dots \times [0, \dots, n_t] &\longrightarrow \mathcal{C}\ell(\mathcal{O}) \\ (e_1, \dots, e_t) &\longrightarrow [\mathfrak{q}_1^{e_1} \cdot \dots \cdot \mathfrak{q}_t^{e_t}]\end{aligned}$$

is injective. Then, the problem should remain difficult.
We can reformulate this in a way that allows $(\bar{\mathfrak{q}}_i, n_i) \in S$:

$$\begin{aligned}[-n_1, \dots, n_1] \times \dots \times [-n_t, \dots, n_t] &\longrightarrow \mathcal{C}\ell(\mathcal{O}) \\ (e_1, \dots, e_t) &\longrightarrow [\mathfrak{q}_1^{e_1} \cdot \dots \cdot \mathfrak{q}_t^{e_t}]\end{aligned}$$

is injective. If $e_i < 0$, then $\mathfrak{q}_i^{e_i}$ corresponds to $(\bar{\mathfrak{q}}_i)^{|e_i|}$.

SECURITY PARAMETERS - FIRST CHOICE

Consider an arbitrary supersingular endomorphism ring $\mathcal{O}_{\mathfrak{B}} \subset \mathfrak{B}$ with discriminant p^2. There is a positive definite rank 3 quadratic form

$$\begin{array}{rcl} \text{disc} : \mathcal{O}_{\mathfrak{B}}/\mathbb{Z} & \longrightarrow & \mathbb{Z} \\ /\!/ \quad\quad \alpha & \longmapsto & |\text{disc}(\alpha)| = |\text{disc}\,(\mathbb{Z}\,[\alpha])\,| \\ \bigwedge^2 (\mathcal{O}_{\mathfrak{B}}) \supseteq \mathbb{Z} \wedge \mathcal{O}_{\mathfrak{B}} & & \end{array}$$

representing discriminants of orders embedding in $\mathcal{O}_{\mathfrak{B}}$.

The general order $\mathcal{O}_{\mathfrak{B}}$ has a reduced basis $1 \wedge \alpha_1, 1 \wedge \alpha_2, 1 \wedge \alpha_3$ satisfying

$$|\text{disc}(1 \wedge \alpha_i)| = \Delta_i \text{ where } \Delta_i \sim p^{2/3}$$

(Minkowski bound: $c_1 p^2 \leq \Delta_1 \Delta_2 \Delta_3 \leq c_2 p^2$).

In order to hide $\mathcal{O}_n$ in $\mathcal{O}_{\mathfrak{B}}$ we impose

$$\ell^{2n} |\Delta_K| > c p^{2/3} \qquad \Rightarrow \qquad n \sim \frac{\log_\ell(p)}{3}$$

so that there is no special imaginary quadratic subring in $\mathcal{O}_{\mathfrak{B}} = \text{End}(E_n)$.

OSIDH - Vortex & Whirlpool

We can read this scheme using the terminology introduced at the beginning.

After the choice of the secret key, we observe a vortex: Alice (respectively Bob) acts on an isogeny crater (that in the case of $\mathcal{O}_K = \mathbb{Z}[\omega]$ or $\mathbb{Z}[i]$ consists of a single points) with the primes $\mathfrak{p}_1^{e_1} \cdot ... \cdot \mathfrak{p}_t^{e_t}$ (respectively $\mathfrak{q}_1^{d_1} \cdot ... \cdot \mathfrak{q}_t^{d_t}$).

This action is eventually transmitted along the ℓ-isogeny chain and we get a whirlpool. We can think of the isogeny volcano as rotating under the action of the secret keys and the initial ℓ-isogeny path transforming into the two secret isogeny chains.

CONCLUSIONS

By imposing the data of an orientation by an imaginary quadratic ring $\mathcal{O}$, we obtain an augmented category of supersingular curves on which the class group $\mathcal{C}\ell(\mathcal{O})$ acts faithfully and transitively.

This idea is already implicit in the CSIDH protocol, in which supersingular curves over $\mathbb{F}_p$ are oriented by the Frobenius subring $\mathbb{Z}[\pi] \cong \mathbb{Z}[\sqrt{-p}]$.

In contrast we consider an elliptic curve E_0 oriented by a CM order $\mathcal{O}_K$ of class number one. To obtain a nontrivial group action, we consider ℓ-isogeny chains, on which the class group of an order $\mathcal{O}$ of large index ℓ^n in $\mathcal{O}_K$ acts.

The map from ℓ-isogeny chains to its terminus forgets the structure of the orientation, and the original curve E_0, giving rise to a generic s.s. elliptic curve.

We define a new oriented supersingular isogeny Diffie-Hellman (OSIDH) protocol, which has fewer restrictions on the proportion of supersingular curves covered and on the torsion group structure of the underlying curves.

Moreover, the group action can be carried out effectively solely on the sequences of moduli points (such as j-invariants) on a modular curve, thereby avoiding expensive isogeny computations, and is further amenable to speedup by precomputations of endomorphisms on the base curve E_0.

This is a work in progress and we still want to develop the following aspects:

- Security analysis and setting security parameters.
- Implementation and algorithmic optimization.
- Use of canonical liftings.

MERCI POUR VOTRE ATTENTION