TERNARY QUADRATIC FORMS OVER LOCAL FIELDS

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1897 K. HENSEL (GERMAN MATHEMATICIAN) INTRODUCED p-ADIC NUMBERS BASED ON THE WORK OF KUMMER. The main motivation was the analogy between $\mathbb{Z}$ together with its field of fractions and $\mathbb{C}[X]$ together with $\mathbb{C}(X)$

$$P(X) \in \mathbb{C}[X], \alpha \in \mathbb{C} \xRightarrow{\text{TAYLOR EXPANSION}} P(X) = a_0 + a_1(X-\alpha) + \dots + a_n(X-\alpha)^n$$

ANALOGOUSLY

$$m \in \mathbb{Z}, p \text{ PRIME} \xRightarrow{\text{BASE } p} m = a_0 + a_1 p + \dots + a_n p^n$$

THIS GIVES LOCAL (around a point) INFORMATION ON $P(X)$ OR m (VANISHING ORDER...)

NOW FOR RATIONAL FUNCTIONS WE CAN EXTEND THE ANALOGY AND CONSIDER FORMAL SUMS

$$f(X) = \frac{P(X)}{Q(X)} = \sum_{i \geq n_0} a_i (X-\alpha)^i \quad \text{Laurent exp.}$$

$$\Rightarrow \mathbb{C}(X) \hookrightarrow \mathbb{C}((X-\alpha))$$

$$\sum_{n \geq n_0} a_n p^n \qquad \mathbb{Q} \hookrightarrow \mathbb{Q}_p$$

$$x \longmapsto p\text{-adic expansion}$$

1918 THE DEFINITION WAS IMPROVED BY OSTROWSKI

HE WAS WORKING WITH ABSOLUTE VALUES ON $\mathbb{Q}$ AND FOUND ANOTHER DESCRIPTION OF $\mathbb{Q}_p$

His work describes the topology that this object have

1944 M. KRASNER INTRODUCED THE CONCEPT OF ULTRAMETRIC FIELDS (TO WHICH p-ADIC NUMBERS BELONG) AND IN 1954 HE PROVED THAT THE TOPOLOGY IS TOTALLY DISCONNECTED

LET p BE A PRIME NUMBER AND LET $n \in \mathbb{Z}$ we can write n in base p

$$n = a_0 + a_1 p + \dots + a_k p^k$$

Definition A P-ADIC INTEGER IS A FORMAL SERIE

$$a = \sum_{i \geq 0} a_i p^i \qquad 0 \leq a_i < p$$

THIS SET IS DENOTED $\mathbb{Z}_p$

OF COURSE WE GET A WELL DEFINED MAP $\mathbb{Z}_p \xrightarrow{\pi_k} \mathbb{Z}/p^k\mathbb{Z}$ OBTAINED BY CUTTING a AT k^{th} TERM

$$\sum_{i \geq 0} a_i p^i \longmapsto a_0 + a_1 p + \dots + a_{k-1} p^{k-1}$$

NOTICE THAT $\pi_k(a) \equiv \pi_{k-1}(a) \mod p^{k-1}$

If sequence $(a_0, \dots, a_n, \dots)$ of integers such that $a_k \equiv a_{k'} \mod k'$ $\forall k' < k$ defines a unique p-adic integer

THUS WE GET THE FOLLOWING BIJECTION $\mathbb{Z}_p \simeq \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$

Recall INVERSE SYSTEM : $(A_i)_{i \in I}$ WITH MAPS $f_{ij} : A_j \longrightarrow A_i$ $\forall i \leq j$ SUCH THAT $f_{ii} = \mathrm{Id}$ AND $f_{ij} \circ f_{jk} = f_{ik}$

$$\varprojlim_{i \in I} A_i = \{ \vec{a} \in \prod_{i \in I} A_i \mid a_i = f_{ij}(a_j) \ \forall i \leq j \text{ in } I \}$$

IN OUR CASE $\mathbb{Z}_p$ IS AN INVERSE LIMIT OF RINGS (COMMUTATIVE WITH 1) ENDOWING $\mathbb{Z}_p$ WITH THE SAME STRUCTURE

Remark We have an obvious inclusion $\mathbb{Z} \hookrightarrow \mathbb{Z}_p$

Definition THE P-ADIC NUMBERS ARE SERIES OF THE FORM

$$\alpha = \sum_{n \geq n_0} a_n p^n$$

THE SET IS DENOTED $\mathbb{Q}_p$ AND IT IS A FIELD $\mathbb{Q}_p = \mathrm{Frac}(\mathbb{Z}_p)$

Of course $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$

PROOF FIRST PROVE THAT $\alpha \in \mathbb{Z}_p^\times \Leftrightarrow \pi_1(\alpha) \neq 0$

this is because $\alpha \in \mathbb{Z}_p^\times \Leftrightarrow \pi_n(\alpha) \in \mathbb{Z}/p^n\mathbb{Z}$ $\forall n$.

SECONDLY, THIS IMPLIES

$$\mathrm{Frac}\, \mathbb{Z}_p = \mathbb{Z}_p[\tfrac{1}{p}] = \mathbb{Q}$$

TO SHOW THAT IT IS A FIELD IT SUFFICES TO SHOW THAT ANY NONZERO $\alpha \in \mathbb{Z}_p$ IS $\alpha = p^n u$ FOR $u \in \mathbb{Z}_p^\times$

P-ADIC VALUATION

LET $\alpha \in \mathbb{Q}\setminus\{0\} \Rightarrow \alpha = p^k \frac{g}{h}$ FOR $k \in \mathbb{Z}$, $(g,p)=(h,p)=1$

WE DEFINE THE VALUATION AT p (P-ADI VALUATION)

$\nu_p(\alpha) = k$ AND P-ADIC ABSOLUTE VALUE $|\alpha|_p = p^{-\nu_p(\alpha)}$

<u>Properties</u> $\nu_p : \mathbb{Q} \longrightarrow \mathbb{Z} \cup \{\infty\}$ $\quad \nu_p(0) = \infty$ by convention

$\nu_p(ab) = \nu_p(a) + \nu_p(b)$

$\nu_p(a+b) \geq \min\{\nu_p(a) + \nu_p(b)\}$

$\nu_p(a) = \infty \iff a = 0$

$|\cdot|_p : \mathbb{Q} \longrightarrow \mathbb{R}_{\geq 0}$

$|ab|_p = |a|_p |b|_p$

$|a+b|_p \leq \max\{|a|_p, |b|_p\} \leq |a|_p + |b|_p$

$|a|_p = 0 \iff a = 0$

Clearly this absolute value introduce a metric

$$d_p(a,b) = |b-a|_p$$

<u>Theorem (Ostrowski)</u> EVERY NONTRIVIAL ABSOLUTE VALUE ON $\mathbb{Q}$ IS EQUIVALENT TO $|\cdot|_p$ OR TO THE USUAL ABSOLUTE VALUE

this means that they define the same topology

$|\,|_1 \sim |\,|_2 \iff \exists \alpha \in \mathbb{R} : |x|_1 = |x|_2^{\alpha} \ \forall x \in K$

<u>Theorem</u> $0 \neq a \in \mathbb{Q}$ THEN $\prod_{\nu} |a|_\nu = 1$ $\quad \nu = \{\infty, 2, 3, 5, 7, \dots\}$

<u>Theorem</u> $\mathbb{Q}_p$ IS THE COMPLETION OF $\mathbb{Q}$ WITH RESPECT TO $|\,|_p$

NON ARCHIMEDEAN FIELDS

<u>Definition</u> AN ABSOLUTE VALUE ON K IS A MAP $|\cdot| : K \longrightarrow \mathbb{R}$ SATISFYING

i) $|x| \geq 0 \ \forall x \in K$, $|x| = 0 \iff x = 0$

ii) $|xy| = |x| \cdot |y| \quad \forall x, y \in K$

iii) $|x+y| \leq |x| + |y| \quad \forall x, y \in K$

WE SAY THAT $|\,|$ ON K IS NON-ARCHIMEDEAN IF, IN ADDITION

iii) $|x+y| \leq \max\{|x|, |y|\} \ \forall x, y \in K$

to it we can associate a valuation $\nu(x) = -\log|x|$ satisfying the same properties as ν_p

A VALUATION IS DISCRETE IF $\nu(K^\times)$ IS DISCRETE IN $\mathbb{R}$

Definition THE VALUATION RING IS $\mathcal{O}_K = \{x \in K \mid |x| \le 1\}$ $\nu(x) \ge 0$
THE VALUATION RING IS A LOCAL DOMAIN WITH A UNIQUE MAXIMAL IDEAL $\mathfrak{p} = \{x \in K \mid |x| < 1\}$ $\nu(x) > 0$ $\Rightarrow \mathcal{O}_K^\times = \{x \in K \mid |x| = 1\}$
THE RESIDUE FIELD IS

$$k = \mathcal{O}_K / \mathfrak{p}$$

$\forall p_1, p_2 \ \exists \ \mathcal{U}_1 \ni p_1, \mathcal{U}_2 \ni p_2 : \mathcal{U}_1 \cap \mathcal{U}_2 = \emptyset$ — each point has a compact neighborhood

Definition A LOCAL FIELD IS A HAUSDORFF, LOCALLY COMPACT FIELD WITH A NON-DISCRETE TOPOLOGY

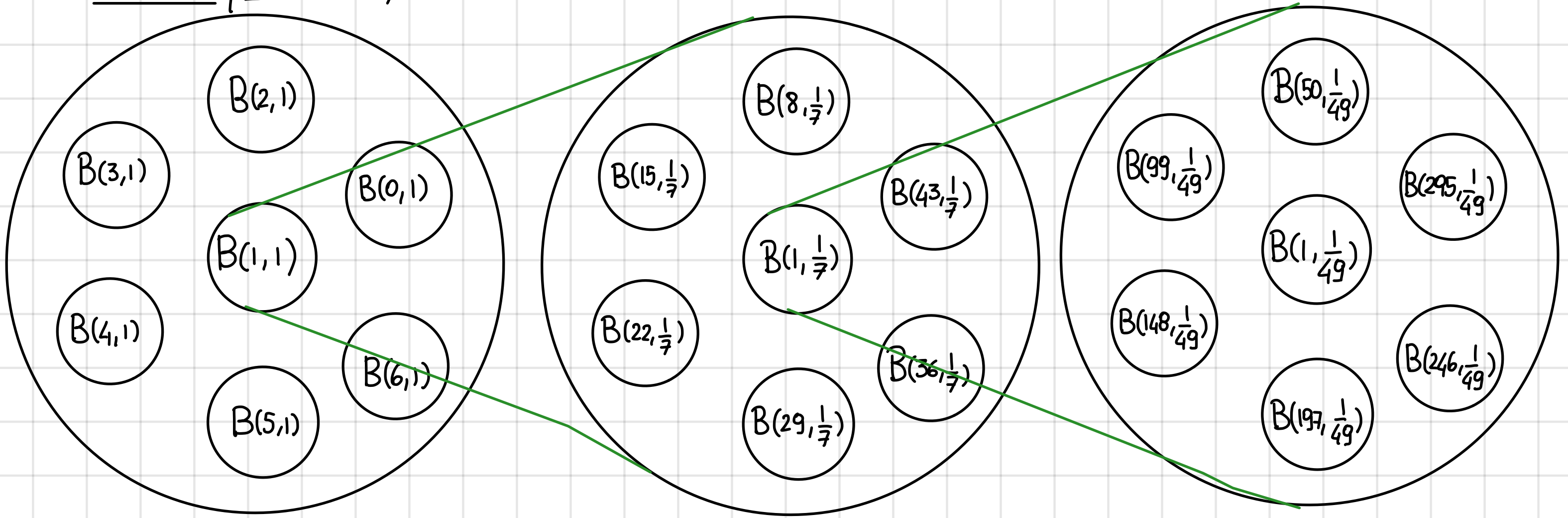
Theorem EVERY LOCAL FIELD IS ISOMORPHIC as a topological field to
1) F ARCHIMEDEAN AND $F = \mathbb{R}$ OR $F = \mathbb{C}$
2) F IS NON-ARCHIMEDEAN AND char $F = 0$ AND F FIN EXT OF $\mathbb{Q}_p$
3) F IS NON-ARCHIMEDEAN AND char $F = p \Rightarrow F$ FIN EXT OF $\mathbb{F}_p((t))$

Lemma F FIELD. THEN THE FOLLOWING STATEMENT HOLD
a) F IS ARCHIMEDEAN LOCAL FIELD $\Leftrightarrow$ F IS COMPLETE WITH RESPECT TO AN ARCHIMEDEAN ABSOLUTE VALUE
b) F IS NON ARCHIMEDEAN $\Leftrightarrow$ F IS COMPLETE WITH RESPECT TO A NON TRIVIAL DISCRETE VALUATION

Lemma F IS NON ARCHIMEDEAN $\Rightarrow$ F IS TOTALLY DISCONNECTED AND $\mathcal{O}_F$ IS COMPACT TOTALLY DISCONNECTED TOPOLOGICAL RING
the connected component containing x is $\{x\}$

Proposition K NON-ARCHIMEDEAN FIELD
1) $b \in B(a,r) \Rightarrow B(a,r) = B(b,r)$
2) $B(a,r)$ IS OPEN AND CLOSED
3) $B(a,r) \cap B(b,s) \neq \emptyset \Leftrightarrow B(a,r) \subset B(b,s)$ OR VICEVERSA

example $\mathbb{Z}_7$



QUATERNION ALGEBRAS AND QUADRATIC FORMS

We now try to classify quaternion algebras over local fields.

Theorem LET $F \neq \underbrace{\mathbb{C}}_{F=\mathbb{C} \text{ IS ARCHIMEDEAN CASE AND WE ALREADY KNOW } B \simeq M_2(\mathbb{C})}$ BE A LOCAL FIELD. THENE THERE IS A UNIQUE DIVISION QUATERNION ALGEBRA B OVER F UP TO F-ALGEBRA ISOMORPHISM

PROVING THIS THEOREM IS EQUIVALENT TO THE FOLLOWING

Proposition $F \neq \mathbb{C}$ LOCAL FIELD. THERE IS A UNIQUE $\overbrace{\text{ANISOTROPIC}}^{h(x)=0 \Leftrightarrow x=0}$ TERNARY QUADRATIC FORM OVER F UP TO SIMILARITY

Lemma A $\overbrace{\text{QUADRATIC SPACE}}^{(V,Q) \text{ A VECTOR SPACE TOGETHER WITH A QUADRATIC FORM}}$ V OVER A FINITE FIELD WITH $\dim_F V \geq 3$ IS ISOTROPIC

Lemma $\mathrm{char}\, k \neq 2$. $Q: M \hookrightarrow \mathcal{O}$ QUADRATIC FORM OVER R. THEN THE REDUCTION $Q \bmod \mathfrak{p}: M \otimes_R k \longrightarrow k$ OF Q MODULO $\mathfrak{p}$ IS NONSINGULAR OVER k. MOREOVER Q IS ISOTROPIC OVER $\mathcal{O} \Leftrightarrow$ $Q \bmod \mathfrak{p}$ IS ISOTROPIC.

Lemma SUPPOSE $\mathrm{char}\, k \neq 0$. $Q: M \longrightarrow \mathcal{O}$ NONSINGULAR WITH M FREE OF RANK $\geq 3 \Rightarrow Q$ IS ISOTROPIC

Lemma $\mathrm{char}\, k \neq 0 \Rightarrow F^\times / F^{\times 2} \simeq (\mathbb{Z}/2\mathbb{Z})^2$ AND IT IS REPRESENTED BY THE CLASSES OF $1, e, \pi, e\pi$ WITH $e \in R^\times$ REDUCES MODULO $\mathfrak{p}$ TO A NONSQUARE IN k

Corollary F NONARCHIMEDEAN LOCAL FIELD WITH VALUATION RING $\mathcal{O}$ AND UNIFORMIZER π. LET B BE A QUATERNION ALGEBRA $/F$

- IF $\mathrm{char}\, k \neq 2 \Rightarrow B$ IS A DIVISION ALGEBRA $\Leftrightarrow$
  $B \simeq (e, \pi \mid F)$
  $e \in R^\times$ IS NON TRIVIAL IN $k^\times / k^{\times 2}$
- IF $\mathrm{char}\, F = \mathrm{char}\, k = 2 \Rightarrow B$ IS A DIVISION ALGEBRA $\Leftrightarrow$
  $B \simeq [b, \pi, F)$
  $b \in R$ NONTRIVIAL IN $k / \wp(k)$

- OVER $\mathbb{Q}_p$ THERE IS A UNIQUE QUATERNION DIVISION ALGEBRA
  $B \simeq (e, p \mid \mathbb{Q}_p)$