TERNARY QUADRATIC FORMS

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Notation F FIELD OF CHARACTERISTIC DIFFERENT FROM 2. $B = \left(\frac{a,b}{F}\right)$

We have seen last week that the reduced norm defines a quadratic form. This quadratic form in 4 variables is too big. Since we already know what happens when we restrict it to F, then we will study its behaviour on the space of pure quaternions

$$B_0 = \{\alpha \in B : \operatorname{trd}(\alpha) = 0\} = \{1\}^{\perp}$$

THIS HAS BASIS i, j, k.

$$\operatorname{nrd}(xi + yj + zk) = -ax^2 - by^2 + abz^2$$

$$\Delta = (-a)(-b)(ab) = a^2 b^2$$

THE DISCRIMINANT IS THEN IN THE TRIVIAL CLASS OF $F^\times / F^{\times 2}$

Goal TO CLASSIFY QUATERNION ALGEBRAS $/F$ UP TO ISOMORPHISM IN TERMS OF THIS QUADRATIC FORM.

For morphisms between quadratic forms one allows either *isometries* (invertible change of variables preserving the quadratic form) or *similarities* (rescaling the form by $\alpha \in F^\times$).

WE WILL SEE THAT THE CLASSIFICATION OF QUATERNION ALGEBRAS CAN BE REPHRASED IN TERMS OF QUADRATIC FORMS

Definition $\alpha \in B$ IS A SCALAR IF $\alpha \in F$ AND IT IS PURE IF $\operatorname{trd}(\alpha) = 0$ the standard involution restricted to B^0 is given by $\bar{\ } : \alpha \mapsto -\alpha \Rightarrow B^0 = -1$ eigenspace for $\bar{\ }$.

THE RESTRICTION OF THE REDUCED NORM TO B^0, $\operatorname{nrd}|_{B^0}$, DEFINES A QUADRATIC FORM IN THREE VARIABLES WHICH IS ISOMETRIC TO $\langle -a, -b, ab \rangle$ FOR $a, b \in F^\times$ SUCH THAT $A \cong \left(\frac{a,b}{F}\right)$

Recall TWO QUADRATIC FORMS $Q : V \longrightarrow F$ AND $Q' : V' \longrightarrow F$ ARE SIMILAR IF THERE EXISTS A COUPLE (f, u) : $f : V \longrightarrow V'$ IS AN F VECTOR SPACE ISOMORPHISM AND $Q'(f(x)) = u\, Q(x)$ $\forall x \in V$

Q AND Q' ARE ISOMETRIC IF $u = 1$

Proposition LET B AND B' BE QUATERNION ALGEBRAS $/F$. TFAE:
(i) B AND B' ARE ISOMORPHIC AS F-ALGEBRAS
(ii) B AND $(B')^{op}$ ARE ISOMORPHIC AS F-ALGEBRAS
(iii) B AND B' ARE ISOMETRIC QUADRATIC SPACES
(iv) B^0 AND B'^0 ARE ISOMETRIC AS QUADRATIC SPACES

Sketch of Proof (i) $\Leftrightarrow$ (ii) TRIVIAL FROM $B \cong B^{op}$ VIA STANDARD INVOLUTION
(i) $\Rightarrow$ (iii) FOLLOWS BY - uniqueness of standard involution
- Reduced norm comes from standard involution.

(iii) $\Rightarrow$ (iv) FOLLOWS FROM WITT CANCELLATION
(iv) $\Rightarrow$ (iii) IS TRIVIAL
(iv) $\Rightarrow$ (i) REQUIRES SOME COMPUTATIONS

Theorem THERE IS AN EQUIVALENCE OF CATEGORIES

Quaternion algebras $/F$ with F-algebra isomorphisms and anti-isomorphisms (these are isomorphisms $B \xrightarrow{\sim} (B')^{op}$) $\longleftrightarrow$ Ternary quadratic forms with discriminant $1 \in F^\times/F^{\times 2}$ with isometries

GIVEN BY THE FUNCTOR $B \longmapsto \mathrm{nrd}|_{B^0}$

Theorem THERE ARE BIJECTIONS (AND THEY ARE FUNCTORIAL WITH RESPECT TO F)
i.e., they are compatible with field extension

$\left\{\begin{matrix}\text{QUATERNION ALGEBRAS} \\ \text{OVER } F \text{ UP TO} \\ \text{ISOMORPHISMS}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{TERNARY QUADRATIC} \\ \text{FORMS } /F \text{ WITH DISCR} \\ 1 \in F^\times/F^{\times 2} \text{ UP TO ISOMETRY}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{NON DEGENERATE} \\ \text{TERNARY Q.F. } /F \\ \text{UP TO SIMILARITY}\end{matrix}\right\}$

Proof I BIJECTION $B \longmapsto \mathrm{nrd}|_{B^0}$
- INJECTIVITY COMES FROM THE EQUIVALENCE (i) $\Leftrightarrow$ (iv) IN THE PROPOSITION
- SURJECTIVITY take a quadratic space (Q, V) with $\mathrm{discr}\, Q = 1 \in F^\times/F^{\times 2}$ by choosing a basis we get $Q \cong_{\text{isometric}} \langle a, -b, c \rangle$, some $a, b, c \in F$, $abc \in F^{\times 2}$

IS A SQUARE

$Q \cong \langle -a, -b, c \rangle \cong \langle -a, -b, ab\frac{c}{ab} \rangle \cong \langle -a, -b, ab \rangle \cong \mathrm{nrd}|_{B^0}$

$abc = h^2 \in F^{\times 2} \Rightarrow c = \frac{h^2}{ab}$
$\Rightarrow \frac{c}{ab} = \frac{h^2}{(ab)^2} \in F^{\times 2}$

WITH $B = (a, b \mid F)$

II BIJECTION: $[Q]_{\text{ISOMETRY}} \xrightarrow{\psi} [Q]_{\text{SIMILARITY}}$

SURJECTIVITY we need to prove that every nondegenerate ternary quadratic form is similar to some form with discriminant 1.

this is, in fact, an equality

$$Q \cong \langle a,b,c\rangle \sim abc\langle a,b,c\rangle \cong \langle a^2bc, ab^2c, abc^2\rangle$$
$$\cong \langle bc, ac, ab\rangle : \operatorname{discr} = (abc)^2 = 1 \in F^\times/F^{\times 2}$$

INJECTIVITY $\psi((Q,V)) = \psi((Q',V')) \Rightarrow Q \overset{\text{similar}}{\sim} Q'$
$\Rightarrow \exists f: V \to V'$ AND $u: Q'(f(x)) = uQ(x))$

$$1 = \operatorname{discr}(Q') = u^3 \operatorname{discr} Q = u \operatorname{discr}(Q) \in F^\times/F^{\times 2}$$
$$\Rightarrow u \in F^{\times}/F^{\times 2} \Rightarrow u = c^2 \quad c \in F^\times$$

$$Q'(c^{-1}f(x)) = c^{-2}Q'(f(x)) = u^{-1}uQ(x) = Q(x)$$
$\Rightarrow Q'$ AND Q ARE ISOMETRIC VIA $c^{-1}f$.

CLIFFORD ALGEBRAS

WE CONSTRUCT NOW A FUNCTORIAL INVERSE TO $B \mapsto \operatorname{nrd}|_{B^0} = Q$

LET $Q: V \to F$ BE A QUADRATIC FORM WITH $\dim_F V = n < \infty$

Proposition THERE EXIST AN F-ALGEBRA $Clf(Q)$ SUCH THAT
(a) $\exists\, i: V \xrightarrow{\text{F-LINEAR}} Clf(Q)$ SUCH THAT $i(x)^2 = Q(x)\ \forall x \in V$
(b) Clf HAS THE FOLLOWING UNIVERSAL PROPERTY. IF A IS AN F-ALGEBRA AND i_A IS SUCH THAT $i_A(x)^2 = Q(x)\ \forall x \in V$ THEN $\exists!\, \psi: Clf(Q) \to A$ HOMOMORPHISM SUCH THAT

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, bend right, "i_A"'] & Clf(Q) \arrow[d, "\psi"] \\ & A \end{tikzcd}$$

COMMUTES. $(Clf(Q), i)$ IS UNIQUE AND i IS INJECTIVE
Clf IS CALLED CLIFFORD ALGEBRA

Proof $\mathrm{Ten}(V) = \bigoplus_{d=0}^{\infty} V^{\otimes d}$ $\quad V^{\otimes 0} = F$

IT HAS A MULTIPLICATION: $x \in V^{\otimes d}$, $y \in V^{\otimes e}$ $\quad x \cdot y = x \otimes y \in V^{\otimes(d+e)}$
THEN $\mathrm{Ten}(V)$ HAS A STRUCTURE OF F-ALGEBRA

NOW $I = \langle x \otimes x - Q(x) \mid x \in V \rangle \subseteq \mathrm{Ten}(V)$ TWO SIDED IDEAL

DEFINE $\mathrm{Clf}(Q) = \mathrm{Ten}(V)/I$. IT IS EASY TO SEE THAT IT SATISFIES THE TWO PROPERTIES

There is a well defined map called the *reversal involution*

$$\mathrm{rev}: \mathrm{Clf}(Q) \longrightarrow \mathrm{Clf}(Q)$$
$$x_1 \otimes \dots \otimes x_r \longmapsto x_r \otimes \dots \otimes x_1$$

A FUNCTOR $\mathcal{F}: \mathcal{C} \rightsquigarrow \mathcal{D}$ INDUCES A MAP
$F_{X,Y}: \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$
A FAITHFUL FUNCTOR IS A FUNCTOR SUCH THAT
THIS MAP IS INJECTIVE (surjectivity is called full functor)

Lemma THE ASSOCIATION $Q \longmapsto \mathrm{Clf}(Q)$ INDUCES A FAITHFUL FUNCTOR

Quadratic forms $/F$ under isometries $\rightsquigarrow$ Finite dimensional F-algebras with involutions up to isomorphism.

example $Q: F \longrightarrow F \quad Q(x) = ax^2 \quad a \in F$
$\mathrm{Clf}(Q) \cong F[x]/(x^2 - a)$

$$\left[\begin{array}{l} a \otimes b = ab\overbrace{(1 \otimes 1)}^{x^2} \\ \bigoplus \text{ only f.m. non-zero} \end{array}\right]$$

$\mathrm{Ten}(F) = F \oplus F \oplus (F \otimes F) \oplus (F \otimes F \otimes F) \oplus \dots$ and since $a_1 \otimes \dots \otimes a_n = a_1 \cdots a_n \, 1 \otimes \dots \otimes 1$, then
$\mathrm{Ten}(F) \simeq F[x]$ Now $I = (1 \otimes 1 - Q(1)) \simeq (x^2 - a)$

Remark $(x+y) \otimes (x+y) - x \otimes x - y \otimes y = Q(x+y) - Q(x) - Q(y)$, THEN
$x \otimes y + y \otimes x = T(x,y) \Rightarrow x \perp y \Leftrightarrow x \otimes y = -y \otimes x$

Remark $\mathrm{Ten}(V)$ HAS A NATURAL $\mathbb{Z}_{\geq 0}$ GRADING BY DEGREE AND BY CONSTRUCTION $\mathrm{Clf}(Q) = \mathrm{Ten}(V)/I$ RETAINS A $\mathbb{Z}/2\mathbb{Z}$ GRADING

$$\mathrm{Clf}(Q) = \underbrace{\mathrm{Clf}^0(Q)}_{\text{even degree algebra}} \oplus \underbrace{\mathrm{Clf}^1(Q)}_{\text{odd degree } \underbrace{\text{bimodule}}_{\text{it means right and left module}}}$$

(IT IS A $\mathrm{Clf}^0(Q)$ BIMODULE)

Remark SUPPOSE $e_1, \dots, e_n$ IS A BASIS FOR V THEN

$$e_1 \otimes \dots \otimes e_n \otimes \underline{\alpha} = e_1 \otimes \dots \otimes e_n \otimes \left(\textstyle\sum \alpha_i e_i\right) = \textstyle\sum_i (e_1 \otimes \dots \otimes e_n \otimes e_i)$$
$$= \textstyle\sum_i (e_1 \otimes \dots \otimes e_i \otimes e_i \otimes \dots \otimes e_n) = \textstyle\sum_i (-1)^{n-i} (e_1 \otimes \dots \otimes (e_i \otimes e_i) \otimes \dots \otimes e_n)$$
$$= \textstyle\sum_i (-1)^{n-i} Q(e_i) \, [e_1 \otimes \dots \otimes \underbrace{\hat{e}_i}_{\text{removed}} \otimes \dots \otimes e_n]$$

Prop 2.14 Atiyah-MacDonald

$\Rightarrow$ IF THE DIMENSION OF V IS n $\Rightarrow$ IN $\mathrm{Clf}(Q)$ THE "DEGREE" IS AT MOST n

$\Rightarrow Clf(Q)$ IS GENERATED BY $\{e_{i_1}\otimes\dots\otimes e_{i_d} \mid 1\le i_1<i_2<\dots<i_d\le n,\ d\le n\}$

$\Rightarrow Clf(Q)$ HAS DIMENSION $\dim_F Clf(Q) = \sum_{d=0}^{n}\binom{n}{d} = 2^n$

example IF $Q=\langle a_1,\dots,a_n\rangle$ IS DIAGONAL IN THE BASIS $\{e_i\}_i$ (⇒ the basis is orthogonal)
THEN
$$(e_{i_1}\otimes\dots\otimes e_{i_d})^2 = \mathrm{sgn}(i_1\dots i_d)Q(e_{i_1})\cdots Q(e_{i_d}) = (-1)^{d-1}a_{i_1}\cdots a_{i_d}$$

Lemma THE ASSOCIATION $Q\longmapsto Clf^0(Q)$ DEFINES A FUNCTOR

$$\text{Quadratic forms}/F \text{ under similarities} \longleftrightarrow \text{Finite dimensional } F\text{-algebras with involutions under isomorphisms}$$

example char $F\neq 2$, V BINARY QUADRATIC SPACE $Q=\langle a,b\rangle$ w.r.t. $\{e_1,e_2\}$
THEN $Clf(Q)\cong F\oplus V\oplus (V\otimes V)/_I \cong F\oplus F\oplus F\oplus (F\oplus F)\otimes(F\oplus F)$

$$\cong F\oplus F\oplus F\oplus (F\otimes F)\oplus(F\otimes F)\oplus(F\otimes F)\oplus(F\otimes F)/I$$
$$\cong F\oplus F\oplus F\oplus \boxed{(F\oplus F\oplus F\oplus F)}/I \simeq F\oplus F\oplus F\oplus F\cong$$
$$\cong F\oplus Fe_1\oplus Fe_2\oplus Fe_1\otimes e_2$$

THIS PART HERE HAS BASIS $\{e_1\otimes e_1; e_1\otimes e_2; e_2\otimes e_1; e_2\otimes e_2\}$ WHICH MODULO I BECOMES $e_1\otimes e_2$ SINCE $e_i\otimes e_i = Q(e_i)$ AND $e_2\otimes e_1 = e_1\otimes e_2 - T(e_1;e_2)$

where multiplication is given by $e_1^2=a$ $e_2^2=b$
$e_1\otimes e_2 = -e_2\otimes e_1$ (here supposing the basis orthogonal since Q is diagonal)

$$Clf(Q)\cong(a,b\mid F)$$

THE REVERSE INVOLUTION INDUCES A NORM ON THE CLIFFORD ALGEBRA AND THE RESTRICTION OF THIS NORM TO $i(V)$ ALWAYS GIVES Q. here we're talking about the reduced norm (coming from quaternions)

Notice that the reverse involution is not the standard involution: IT FIXES e_1 AND e_2 AND ACTS AS THE STANDARD INVOLUTION ON $Clf^0(Q)$. In fact, the quadratic form Q on V is not the restriction of the norm of A to $i(V)$ (they differ by a sign)

example NOW LET $Q=\langle a,b,c\rangle$ THEN

$$Clf(Q)\cong F\oplus V\oplus(V\otimes V)\oplus(V\otimes V\otimes V)/I$$

$$Clf^0(Q)\cong F\oplus(V\otimes V)\cong F\oplus(F\oplus F\oplus F)\otimes(F\oplus F\oplus F)/I^0$$
$$\cong F\oplus(F\otimes F)\oplus(F\otimes F)\oplus(F\otimes F)$$
$$\oplus(F\otimes F)\oplus(F\otimes F)\oplus(F\otimes F)$$
$$\oplus(F\otimes F)\oplus(F\otimes F)\oplus(F\otimes F)/I^0$$
$$\cong F\oplus F e_1\otimes e_2\oplus F e_1\otimes e_3\oplus F e_2\otimes e_3$$

WHERE $(e_1\otimes e_2)\otimes(e_1\otimes e_3) = e_1\otimes e_2\otimes e_1\otimes e_3 =$
$= -(e_1\otimes e_1)\otimes e_2\otimes e_3 = -a\,(e_2\otimes e_3)$

$\Rightarrow Clf^0(Q) \cong F\oplus Fi\oplus Fj\oplus Fij \qquad i^2=-ab \quad j^2=-bc$
$\qquad ij=-ji$

$\Rightarrow Clf^0(Q) \cong (-ab,-bc\,|\,F) := B$

Now the reversal involution is the standard involution on $Clf^0(Q)$.

$$\mathrm{nrd}|_{B^0} = \langle ab, bc, ac\rangle \cong \langle abc^2, a^2bc, ab^2c\rangle \cong abc\langle a,b,c\rangle$$

SO IF $\mathrm{discr}\, Q = abc \in F^{\times 2} \Rightarrow \mathrm{nrd}|_{B^0} \cong Q$.

and this gives another proof of the theorem with the bijections.

SPLITTING

Classifying quaternion algebras depends on the theory of ternary quadratic forms

Definition THE HYPERBOLIC PLANE IS THE QUADRATIC FORM $H: F^2 \to F$, $H(x,y)=xy$. A quadratic form is Hyperbolic if $Q\cong H$

Lemma SUPPOSE Q IS NON-DEGENERATE. THEN Q IS ISOTROPIC (represents 0 non trivially) $\Leftrightarrow \exists$ AN ISOMETRY $Q\cong H\perp Q'$ WITH Q' NONDEGENERATE AND H AN HYPERBOLIC PLANE. ↳ orthogonal sum $H\perp Q': V\oplus V \to F$, $(x,y)\mapsto H(x)+Q'(y)$

Lemma Q NON DEGENERATE, $a\in F^\times$. TFAE
a) Q REPRESENTS a
b) $Q\cong\langle a\rangle\perp Q'$ SOME Q' NON DEGENERATE
c) $\underbrace{\langle -a\rangle}_{\text{Form } -ax^2}\perp Q$ IS ISOTROPIC

Theorem LET $B=(a,b\,|\,F)$. TFAE
1. $B\cong(1,1\,|\,F)\cong M_2(F)$
2. B IS NOT A DIVISION RING
3. $\mathrm{nrd}\cong\langle 1,-a,-b,ab\rangle$ IS ISOTROPIC
4. $\mathrm{nrd}|_{B^0}\cong\langle -a,-b,ab\rangle$ IS ISOTROPIC
5. $\langle a,b\rangle$ REPRESENTS 1
6. $b\in \mathrm{Nm}_{K|F}(K^\times)$ WHERE $K=F(i)$

Definition A QUATERNION ALGEBRA B/F IS SPLIT IF $B \simeq M_2(F)$.
A FIELD K CONTAINING F IS SPLITTING FIELD FOR B IF $B \otimes_F K$ IS SPLIT. (example: $\mathbb{H}/\mathbb{C}$).

Lemma K/F QUADRATIC EXTENSION. K IS A SPLITTING FIELD FOR B IF AND ONLY IF $\exists$ $K \hookrightarrow B$ INJECTIVE F-ALGEBRA HOMOMORPHISM

example IF $B = \left(\frac{a,b}{F}\right)$ THEN EITHER $a \in F^{\times 2}$ AND $B \simeq \left(\frac{1,b}{F}\right) \simeq M_2(F)$ IS SPLIT OR $a \notin F^{\times 2}$ AND $K = F(\sqrt{a})$ IS A SPLITTING FIELD FOR B

CONICS & EMBEDDING

Definition A CONIC $\mathcal{C} \subset \mathbb{P}^2(F)$ IS A NONSINGULAR PLANE CURVE OF DEGREE 2. $\mathcal{C}$ AND $\mathcal{C}'$ ARE ISOMORPHIC OVER F IF THERE EXISTS $f \in PGL_3(F) = AUT(\mathbb{P}^2)(F)$ INDUCING $f : \mathcal{C} \xrightarrow{\sim} \mathcal{C}'$

If we identify $\mathbb{P}(B^0) = (B^0 \setminus \{0\})/F^\times \simeq \mathbb{P}^2(F)$ with (the points of) the projective plane over F, then the vanishing locus $\mathcal{C} = V(Q)$ of $Q = \mathrm{nrd}|_{B^0}$ defines a conic over F

$$Q(x,y,z) = \mathrm{nrd}(xi + yj + zij) = -ax^2 - by^2 + abz^2 = 0$$

Corollary THE MAP $B \longrightarrow C = V(\mathrm{nrd}|_{B^0})$ YIELDS A BIJECTION

$$\left\{ \begin{array}{l} \text{Quaternion algebras over } F \\ \text{up to isomorphism} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{Conics over } F \text{ up} \\ \text{to isomorphism} \end{array} \right\}$$

Theorem TFAE
1) $B \simeq M_2(F)$
2) $\mathcal{C}$ HAS AN F-RATIONAL POINT

HYLBERT SYMBOLS

Definition WE DEFINE THE HILBERT SYMBOL

$$(\,,\,)_F : F^\times \times F^\times \longrightarrow \{\pm 1\}$$

BY THE CONDITION THAT $(a,b) = 1 \iff (a,b \mid F) \simeq M_2(F)$ IS SPLIT

We get $(a,b)_F = 1 \iff ax^2 + by^2 = 1$ HAS A SOLUTION : Hilbert criterion for the splitting of a quaternion algebra.

THE HILBERT SYMBOL IS A WELL DEFINED MAP

Lemma LET $a, b \in F^\times$

$$\left.\begin{aligned}
&1.\ (ac^2, bd^2)_F = (a,b)_F \quad \forall\, c,d \in F^\times\\
&2.\ (b,a)_F = (a,b)_F\\
&3.\ (a,b)_F = (a,-ab)_F = (b,-ab)_F
\end{aligned}\right\}\ \text{exercise 2.4}$$

4. $(1,a)_F = (a,-a)_F = 1$
5. IF $a \neq 1$ THEN $(a, 1-a)_F = 1$
6. $\sigma \in \mathrm{AUT}(F)$: $(a,b)_F = (\sigma(a), \sigma(b))_F$

exercises 5/8/14/18