In the previous chapter we have seen that

$$p = x^2 + ny^2 \iff \begin{cases} (-n/p) = 1 \quad \text{and} \\ f_n(x) \equiv 0 \pmod p \text{ has an integer solution} \end{cases}$$

The key ingredient are the polynomials f_n which we know are the minimal polynomials of the primitive elements of the ring class field of $\mathbb{Z}[\sqrt{-n}]$. But how can we find these primitive elements.

DEFINITION WE DEFINE A LATTICE TO BE AN ADDITIVE SUBGROUP $\Lambda \subseteq \mathbb{C}$ GENERATED BY TWO COMPLEX NUMBERS ω_1 AND ω_2 LINEARLY INDEPENDENT OVER $\mathbb{R}$

WE SAY THAT TWO LATTICES ARE HOMOTHETIC IF AND ONLY IF THERE EXISTS AN ELEMENT $c \in \mathbb{C}^\times$ SUCH THAT $\Lambda_1 = c\Lambda_2$. we will care of Λ only up to homothety. Thus, we can normalize and get

$$\frac{1}{\omega_2}\Lambda = \frac{\omega_1}{\omega_2}\mathbb{Z} + \mathbb{Z}$$

usually we choose a positive orientation, i.e., $\operatorname{Im}(\omega_1/\omega_2) > 0$ which suggest looking at the upper half plane $\mathcal{H} = \{\tau \in \mathbb{C} \mid \operatorname{Im}(\tau) > 0\}$

WE GET A SURJECTIVE MAP $\mathcal{H} \longrightarrow \mathcal{L}/\mathbb{C}^\times$ (→ lattices). HOWEVER THIS IS NOT INJECTIVE

Lemma $\forall a,b,c,d \in \mathbb{R}$, $\tau \in \mathbb{C} \setminus \mathbb{R}$, THEN $\operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{(ad-bc)}{|c\tau+d|^2}\operatorname{Im}(\tau)$

Proof $$\frac{a\tau+b}{c\tau+d} = \frac{(ac|\tau|^2 + (ad+bc)s + bd) + ((ad-bc)t)i}{|c\tau+d|^2} \qquad \tau = s+it$$

The ambiguity in associating a τ to a lattice lies in choosing an oriented basis for Λ. suppose

$$\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 = \mathbb{Z}\omega_1' + \mathbb{Z}\omega_2'$$

then $\exists\, a,b,c,d,a',b',c',d' \in \mathbb{Z}$:

$$\begin{cases} \omega_1' = a\omega_1 + b\omega_2 \\ \omega_2' = c\omega_1 + d\omega_2 \end{cases} \qquad \begin{cases} \omega_1 = a'\omega_1' + b'\omega_2' \\ \omega_2 = c'\omega_1' + d'\omega_2' \end{cases}$$

and clearly $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Further,

$$0 < \operatorname{Im}\left(\frac{\omega_1'}{\omega_2'}\right) = \operatorname{Im}\left(\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}\right) \underset{\tau = \frac{\omega_1}{\omega_2}}{=} \operatorname{Im}\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{(ad-bc)}{|c\tau+d|^2}\operatorname{Im}(\tau) \Rightarrow \operatorname{Im}(\tau) > 0$$

IN OTHER WORDS, $\Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) = \{\Gamma \in M_2(\mathbb{Z}) \mid \det \Gamma = 1\}$

Lemma (Binet) (i) TWO ORIENTED BASIS FOR A LATTICE Λ ARE RELATED BY A MATRIX IN $SL_2(\mathbb{Z})$

(ii) Λ_{τ_1} IS HOMOTHETIC TO $\Lambda_{\tau_2} \iff \exists\, \Gamma \in SL_2(\mathbb{Z})$ SUCH THAT $\tau_2 = \Gamma \cdot \tau_1 = (a\tau_1 + b)/(c\tau_2 + d)$

(iii) $\forall \Lambda \in \mathcal{L}\ \exists\, \tau \in \mathcal{H} : \Lambda$ IS HOMOTHETIC TO Λ_τ

we can therefore define an action of $SL_2(\mathbb{Z})$ on $\mathcal{H}$ and obtain a bijection

$$SL_2(\mathbb{Z}) \backslash \mathcal{H} \longleftrightarrow \mathcal{L}/\mathbb{C}^\times$$

Definition $\Gamma(1) = SL_2(\mathbb{Z}) / \{\pm I\}$

Corollary $\Gamma(1)$ ACTS FAITHFULLY ON $\mathcal{H}$

WE NOTE THAT $\Gamma(1)$ CAN BE GENERATED BY TWO IMPORTANT ELEMENTS

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \Gamma(1) = \langle S, T \mid S^2 = (ST)^3 = 1 \rangle$$

$$\tau \mapsto -1/\tau \qquad\qquad \tau \mapsto \tau + 1$$

Definition AN ELLIPTIC FUNCTION FOR Λ IS A FUNCTION $f : \mathbb{C} \longrightarrow \mathbb{C}$ SUCH THAT
1. f IS MEROMORPHIC ON $\mathbb{C}$ (holomorphic on $\mathbb{C} \setminus \{$discrete set of pts$\}$)
2. $f(z + w) = f(z) \quad \forall w \in \Lambda$

THUS, AN ELLIPTIC FUNCTION IS A DOUBLY PERIODIC MEROMORPHIC FUNCTION
there are two complex numbers w_1, w_2 such that

$$f(z + w_1) = f(z + w_2) = f(z) \quad \forall z \in \mathbb{C}$$

WEIERSTRASS $\wp$-FUNCTION

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-w)^2} + \frac{1}{w^2} \right)$$

Theorem
I. $\wp(z)$ IS AN ELLIPTIC FUNCTION FOR Λ WITH DOUBLE POLES ON Λ

II. $\wp(z)$ SATISFIES $\wp'(z)^2 = 4\wp(z)^3 - g_2(\Lambda)\wp(z) - g_3(\Lambda)$

WHERE $g_2(\Lambda) = 60 \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{w^4} \qquad g_3(\Lambda) = 140 \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{w^6}$

III. $\wp(z + w) = -\wp(z) - \wp(w) + \frac{1}{4} \left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)} \right)^2$
$\forall z, w \notin \Lambda$ S.T. $z + w \notin \Lambda$

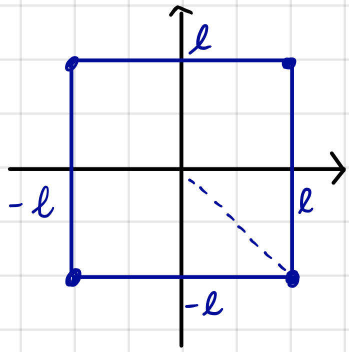
Lemma $G_r(\Lambda) = \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{w^r}$ CONVERGES ABSOLUTELY $\forall r \geq 3$

Proof LET w_1, w_2 BE A BASIS FOR Λ. FOR $\ell \in \mathbb{Z}_{\geq 1}$

$$P(\ell) = \{ a_1 w_1 + a_2 w_2 \mid a_i \in \mathbb{Z}, \max_{i=1,2} |a_i| = \ell \}$$

$\# P(\ell) = 8\ell$

$\operatorname{dist}(0, P(\ell)) \geq \ell \cdot \underbrace{\operatorname{dist}(0, P(1))}_{\delta}$

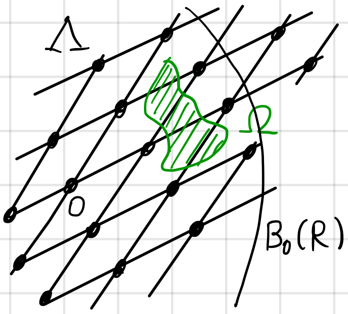


$$\sum_{w \in P(\ell) \cap \Lambda} \frac{1}{|w|^r} \leq \frac{8\ell}{\delta^n \ell^n}$$

$$\sum_{w \in \Lambda - \{0\}} \frac{1}{|w|^r} \leq \sum_{\ell=1}^{\infty} \frac{8\ell}{\delta^n \ell^n} = \frac{8}{\delta^n} \sum_{n=1}^{\infty} \frac{1}{\ell^{n-1}} < \infty$$

Remark Observe $G_{2k+1}(\Lambda) = 0$

Proof of Theorem 1. $\wp(z)$ IS HOLOMORPHIC OUTSIDE Λ
We need to show that its series converges absolutely and uniformly in compact sets Ω



$R \in \mathbb{R}$: $|z| \leq R \;\forall z \in \Omega$

$z \in \Omega$ $w \in \Lambda$ we suppose $2R \leq |w| \Rightarrow |z - w| \geq \frac{|w|}{2}$

$$\left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| = \left| \frac{z(2w - z)}{w^2 (z-w)^2} \right| \leq \frac{R(2|w| + \frac{1}{2}|w|)}{|w|^2 (\frac{1}{2}|w|^2)} = \frac{5R}{|w|^3}$$

THUS $\displaystyle\sum_{w \in \Lambda \setminus \{0\}} \left| \frac{1}{(z-w)^2} - \frac{1}{w^2} \right| \leq 5R \sum_{w \in \Lambda \setminus \{0\}} \frac{1}{|w|^3} < \infty$

$\wp$ CONVERGES ABSOLUTELY AND UNIFORMLY.

$\displaystyle \wp'(z) = -2 \sum_{w \in \Lambda} \frac{1}{(z-w)^3}$ As above, this series converges and it is an elliptic function as well

SINCE $\wp'$ IS PERIODIC,
$\wp(z)$ AND $\wp(z+w)$ HAVE THE SAME DERIVATIVE AND SO

$$\wp(z+w) = \wp(z) + C_1$$

evaluating at $-\frac{\omega}{2}$

$$\wp\left(\frac{w}{2}\right) = \wp\left(-\frac{w}{2}\right) + C_1$$ BUT $\wp$ IS EVEN $\Rightarrow C_1 = 0$

Lemma THE LAURENT SERIES FOR $\wp(z)$ AROUND $z=0$ IS

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{+\infty} (2n+1)\, G_{2n+2}(\Lambda)\, z^{2n}$$

Proof For $k>1$ $|z|<|w|$

$$\frac{1}{(z-w)^2} - \frac{1}{w^2} = \frac{1}{w^2}\left(\frac{1}{(1-\frac{z}{w})^2} - 1\right) \overset{\text{AROUND } 1}{=} \frac{1}{w^2} \sum_{n\geq 1} (n+1) \frac{z^n}{w^n}$$

$$\wp(z) = \frac{1}{z^2} + \sum_{n\geq 1} \sum_{w\in\Lambda\setminus\{0\}} (n+1) \frac{z^n}{w^{2+n}} = \frac{1}{z^2} + \sum_{n\geq 1} (n+1)\, G_{n+2}\, z^n$$

We observe that $G_{2n+1}=0$. then

$$\wp(z) = \frac{1}{z^2} + \sum_{n=1}^{+\infty} (2n+1) G_{2n+2} z^{2n} = \cancel{\frac{1}{z^2}} + 3G_4 z^2 + 5G_6 z^4 + 7G_8 z^6 + \dots$$

$$\wp'(z) = -\frac{2}{z^3} + \sum_{n=1}^{+\infty} 2n(2n+1) G_{2n+2} z^{2n-1} = -\frac{2}{z^3} + 6G_4 z + 20 G_6 z^3 + 42 G_8 z^5 + \dots$$

$$\wp(z)^3 = \cancel{\frac{1}{z^6}} + \cancel{9G_4 \frac{1}{z^2}} + \cancel{15} G_6 + 21 G_8 z^2 + \dots$$

$$\wp'(z)^2 = \cancel{\frac{4}{z^6}} - \cancel{24 G_4 \frac{1}{z^2}} - \cancel{80} G_6 - 168 G_8 z^2 + \dots$$

$$\wp''(z) = \frac{6}{z^4} + \sum_{n=1}^{\infty} (2n-1)2n(2n+1) G_{2n+2} z^{2n-2} = \frac{6}{z^4} + 6G_4 + 60 G_6 z^2 + 210 G_8 z^4 + \dots$$

$$\wp(z)^2 = \frac{1}{z^4} + 6G_4 + 10 G_6 z^2 + (9G_4 + 14 G_8) z^4 + \dots$$

And we may guess that

$$\wp'(z)^2 \approx 4\wp(z)^3 - 60 G_4 \wp(z) - 140 G_6$$

We define $f(z) = \wp'(z)^2 - 4\wp(z)^3 + 60 G_4 \wp(z) - 140 G_6$
IT IS EASY TO SEE THAT $f(0)=0$ AND $f(w)=0$ $\forall w\in\Lambda$ FOR THE PERIODICITY. SINCE f IS HOLOMORPHIC $\Rightarrow$ f IS CONSTANT $\Rightarrow$ $f=0$

Theorem THE FIELD OF DOUBLY PERIODIC MEROMORPHIC FUNCTIONS FOR Λ IS $\mathbb{C}(\wp,\wp')$ AND WE HAVE AN ISOMORPHISM

$$\frac{\mathbb{C}[X,Y]}{(Y^2-4X^3+g_4X+g_6)} \xrightarrow{\sim} \mathbb{C}[\wp,\wp']$$

example $\wp''(z) = 6\wp'(z)^2 - 30 G_4$

Lemma EACH G_{2n} IS A POLYNOMIAL IN G_4 AND G_6 WITH RATIONAL COEFFICIENTS

Proof BY INDUCTION

- $G_8 = \frac{3}{7} G_4^2$ From $\wp''(z) = 6\wp(z)^2 - 30 G_4$
- SUPPOSE IT TRUE FOR G_{2k} $\forall k \leq n$

Observe that the coefficient of z^{2n-2} in $\wp''(z)$ is

$$(2n-1)2n(2n+1)G_{2n+2}$$

and in $6\wp(z)^2$ is a rational polynomial in G_{2k} $k=2,\dots,n+1$

$$(2n+1)2n(2n-1)G_{2n+2} = 6P(G_4,\dots,G_{2n+2})$$

Notice that the coefficient of G_{2n+2} in the polynomial is $2(2n+1)$ - DOUBLE PRODUCT BETWEEN $(2n+1)G_{2n+2}$ AND $\frac{1}{z^2}$

$$\Rightarrow \underbrace{\left((2n+1)2n(2n-1) - 12(2n+1)\right)}_{\neq 0 \ \forall n \geq 2} G_{2n+2} = 6P(G_4,\dots,G_{2n})$$

Theorem $\Lambda_t = \pi\mathbb{Z} + it\mathbb{Z}$ WHERE $t \in \mathbb{R}$

$$\lim_{t\to+\infty} G_{2n}(\Lambda_t) = 2\frac{\zeta(2n)}{\pi^{2n}}$$

Proof $$G_{2n}(\Lambda_t) = \sum_{w\in\Lambda\setminus\{0\}} \frac{1}{w^{2n}} = \sum_{(z_1,z_2)\in\mathbb{Z}^2\setminus\{0\}} \frac{1}{(z_1\pi + z_2 ti)^{2n}}$$

$$= \sum_{z_1\in\mathbb{Z}\setminus\{0\}} \frac{1}{(z_1\pi)^{2n}} + \sum_{\substack{z_2\in\mathbb{Z} \\ z_1\in\mathbb{Z}\setminus\{0\}}} \frac{1}{(z_1\pi + z_2 ti)^{2n}} = 2\sum_{z_1\geq 1} \frac{1}{(z_1\pi)^{2n}} + \dots$$

$$= 2\frac{\zeta(2n)}{\pi^{2n}} + \dots$$

and now taking the limit we cancel the second term. notice that we can switch the limit and the sum because the normal convergence of G_{2n}.

Corollary $\zeta(2n)/\pi^{2n} \in \mathbb{Q}$ $\forall n \geq 1$

Proof WE PROVE FIRST THE CASE $n = 1,2,3$

It can be proved that $\sin(\pi z) = \pi z \prod_{n=1}^{+\infty}\left(1 - \frac{z^2}{n^2}\right) = \pi z - \pi z^3 \zeta(2) + \dots$

On the other hand

$$\sin(\pi z) = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}(\pi z)^{2n-1}}{(2n-1)!} = \pi z - \frac{(\pi z)^3}{3!} + \dots$$

COMPARING COEFFICIENTS OF z^3:

$$\pi\zeta(2)=\frac{\pi^3}{6} \Rightarrow \zeta(2)=\frac{\pi^2}{6}$$

SAME THING FOR $\zeta(4)=\frac{\pi^4}{90}$ AND $\zeta(6)=\frac{\pi^6}{945}$

NOW $\frac{2\zeta(2n)}{\pi^{2n}}=\lim_{t\to+\infty} G_{2n}(\Lambda_t)=\lim_{t\to+\infty} P(G_2(\Lambda_t),G_4(\Lambda_t))$

$=P(\lim_{t\to+\infty} G_2(\Lambda_t),\lim_{t\to+\infty} G_4(\Lambda_t))=P\left(\frac{2\zeta(4)}{\pi^4},\frac{2\zeta(6)}{\pi^6}\right)$

$=P\left(\frac{2}{90},\frac{2}{945}\right)\in\mathbb{Q}$ SINCE P IS RATIONAL.

II APPROACH IT RELIES ON THE FOLLOWING FUNCTIONAL EQUATION FOR ζ

$$\zeta(z)=2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin\frac{\pi z}{2}$$
$$\zeta(1-z)=2(2\pi)^{-z}\Gamma(z)\zeta(z)\cos\frac{\pi z}{2}$$

taking $z=2n$

$$\zeta(1-2n)=2(2\pi)^{-2n}\Gamma(2n)\zeta(2n)\cos\frac{2\pi n}{2}$$

$$2^{2n}\zeta(1-2n)=\frac{2\zeta(2n)}{\pi^{2n}}\underbrace{\Gamma(2n)}_{\text{recall }\Gamma(n+1)=n!\ \forall n\in\mathbb{Z}_{>0}}\cos\pi n$$

$$\frac{\zeta(2n)}{\pi^{2n}}=\underbrace{\frac{2^{2n-1}}{(2n-1)!\cos(\pi n)}}_{\mathbb{Q}}\zeta(1-2n)$$

Now we know that $\zeta(-n)=-\frac{B_{n+1}}{n+1}$ $\forall n\in\mathbb{Z}_{>0}$
where B_n are Bernoulli's Numbers

$$B_n=\sum_{k=0}^{n}\binom{n}{k}B_k \qquad B_0=1$$

$\Rightarrow \zeta(1-2n)\in\mathbb{Q}$.

Remark IT CAN BE PROVEN THAT

$$G_{2k}(\tau)=2\zeta(2k)+2\frac{(2\pi i)^k}{(2k-1)!}\sum_{n=1}^{\infty}\overbrace{\sigma_{2k-1}(n)}^{\sum_{d\mid n}d^{2k-1}}q^n \qquad q=e^{2\pi i\tau}$$

Weierstrass ζ-function

As can be seen from the Laurent expansion of the Weierstrass $\wp$-function, it has double poles with vanishing residues at the points of Λ $\Rightarrow$ we can integrate

Theorem $\zeta(z,\Lambda)=\frac{1}{z}+\sum_{w\in\Lambda\setminus\{0\}}\left(\frac{1}{z-w}+\frac{1}{w}+\frac{z}{w^2}\right)$ IS AN ABSOLUTELY AND UNIFORMLY CONVERGENT SERIES ON COMPACT SUBSETS OF $\mathbb{C}\setminus\Lambda$ AND

I. $\frac{d}{dz}\zeta(z)=-\wp(z)$

II. $\zeta(z+w)=\zeta(z)+\eta(w)$ where $\eta:\Lambda\longrightarrow\mathbb{C}$ is a homomorphism

III. $\tau\eta(1)=\eta(\tau)+2\pi i$

IV. $\int\wp(z)^2\,dz=\frac{1}{6}\wp'(z)+5G_4z+C$

$$\int\wp(z)^3\,dz=\frac{1}{120}\wp'''(z)-9G_4\zeta(z)+14G_6z+C$$

WHERE η IS CALLED THE QUASI-PERIOD OF ζ AND IT MEASURE THE FAILURE OF ζ TO BE AN ELLIPTIC FUNCTION.

Notice that η encodes the periods of differentials of E_Λ when paired with a basis of the singular Homology (later more details)

AROUND 0 $\quad\zeta(z)=\frac{1}{z}-\sum_{n\geq1}G_{2n+2}z^{2n+1}$

Weierstrass σ-Function

$$\sigma(z)=z\prod_{w\in\Lambda\setminus\{0\}}\left(1-\frac{z}{w}\right)\exp\left(\frac{z}{w}+\frac{z^2}{2w^2}\right)$$

Theorem THIS INFINITE PRODUCT CONVERGES ABSOLUTELY TO A HOLOMORPHIC FUNCTION ON $\mathbb{C}$ WITH SIMPLE ZEROES ON Λ AND NO-ZEROES ELSEWHERE

$$\sigma(z+w)=\sigma(z)\cdot\psi(w)\cdot\exp\left(\eta(w)\left(z+\frac{w}{2}\right)\right)$$

where $\psi(w)=\begin{cases}1 & w\in2\Lambda\\ -1 & \text{otherwise}\end{cases}$

Theorem $d\log\sigma(z)=\zeta(z)$ logarithmic derivative

$$\wp(z_1)-\wp(z_2)=\frac{\sigma(z_1+z_2)\,\sigma(z_1-z_2)}{\sigma(z_1)^2\,\sigma(z_2)^2}$$

j-function

Remark If F is an elliptic function for $\Lambda \Rightarrow F(\lambda^{-1}z)$ is an elliptic function for $\lambda\Lambda$

$$\wp(\lambda z, \lambda\Lambda) = \frac{1}{\lambda^2 z^2} + \sum_{w \in \Lambda\setminus\{0\}} \left(\frac{1}{(\lambda z - \lambda w)^2} - \frac{1}{\lambda^2 w^2} \right) = \lambda^{-2}\wp(z, \Lambda)$$

Definition GIVEN Λ, WE WRITE $\Delta(\Lambda) = g_2^3 - 27 g_3^2$. THIS IS CLOSELY RELATED TO THE DISCRIMINANT OF THE CUBIC POLYNOMIAL THAT APPEARS IN THE DIFFERENTIAL EQUATION FOR $\wp$.
IF $e_1; e_2; e_3$ ARE ROOTS OF SUCH A POLYNOMIAL,

$$\Delta(\Lambda) = 16(e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2$$

$\hookrightarrow e_1 = \wp_1\left(\frac{w_1}{2}\right) \quad e_2 = \wp_2\left(\frac{w_2}{2}\right) \quad e_3 = \wp\left(\frac{w_1 + w_2}{2}\right)$

Proposition FOR A LATTICE Λ, $\Delta(\Lambda) \neq 0$.

WE DEFINE THE j-INVARIANT OF A LATTICE IS

$$j(\Lambda) = 1728 \frac{g_2(\Lambda)^3}{g_2(\Lambda)^3 - 27 g_3(\Lambda)^2}$$

Nice fact about j-invariant is that it characterize lattices up to homothety:

Theorem IF Λ AND Λ' ARE TWO LATTICES IN $\mathbb{C}$, THEN $j(\Lambda) = j(\Lambda') \Leftrightarrow \Lambda \sim \Lambda'$

Remark $\Delta(T \cdot z) = (cz + d)^{12}\Delta(z) \quad \forall$ any $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$
infact $g_2(T \cdot z) = (cz + d)^4 g_2(z)$
$g_3(T \cdot z) = (cz + d)^6 g_3(z)$

There is another definition for the j-invariant. We'll present it here since j invariant plays a central role.
WE FIRST DEFINE THE DEDEKIND η-FUNCTION

$$\tau \in \mathcal{H} \qquad \eta(\tau) = q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n)$$

THIS FUNCTION SATISFIES $\eta(\tau + 1) = \zeta_{24}\, \eta(\tau) \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\, \eta(\tau)$

where we choose the branch of the square root which is positive on the positive real axis.

THEN THE MODULAR DISCRIMINANT $\Delta(\tau)=\eta(\tau)^{24}$ IS NOW CLEARLY A MODULAR FORM OF WEIGHT 12 ON $SL_2(\mathbb{Z})$
WE DEFINE THE j-INVARIANT TO BE

$$j(\tau)=\frac{E_4^3(\tau)}{\Delta(\tau)}$$

$$E_4(q)=1+240\sum_{n\geq 1}\sigma_3(n)q^n \qquad \sigma_3(n)=\sum_{d\mid n}d^3$$

Normalized Eisenstein series. THE j-INVARIANT HAS FOURIER EXPANSION

$$j(q)=\frac{1}{q}\left(1+240\sum_{n\geq 1}n^3\frac{q^n}{1-q^n}\right)\prod_{n\geq 1}(1-q^n)^{24}=\frac{1}{q}+744+196884q+21493760q^2$$

Clearly $j(z)$ is a holomorphic function on $\mathcal{H}$ having meromorphic continuation to the cusps and it is $SL_2(\mathbb{Z})$-invariant
However, j is not an algebraic function in the sense that
$\sigma\in Aut(\mathbb{C})$, $j(\tau)^\sigma\neq j(\tau^\sigma)$

Weber Function

THE MODULAR DISCRIMINANT DO NOT VANISH ON THE UPPER HALF PLANE, SO WE MAY CHOOSE A HOLOMORPHIC CUBIC ROOT WHICH IS REAL VALUED ALLOWING US TO DEFINE

$$\gamma_2(q)=\frac{E_4(q)}{\Delta(q)^{1/3}}$$

WHICH IS HOLOMORPHIC ON $\mathbb{H}$ AND HAS TRANSFORMATION PROPERTIES

$$\begin{cases}\gamma_2(z+1)=\zeta_3^2\,\gamma_2(z)\\ \gamma_2(-\frac{1}{z})=\gamma_2(z)\end{cases}$$

BY INDUCTION $\gamma_2\left(\dfrac{az+b}{cz+d}\right)=\zeta_3^{ac-bd+a^2cd-cd}\,\gamma_2(z)$

From this we see that γ_2 is a modular function for the congruence subgroup:

$$\Gamma_0^0(3)^+=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})\ \middle|\ \begin{array}{l}a\equiv d\equiv 0 \bmod 3\ \text{OR}\\ c\equiv b\equiv 0 \bmod 3\end{array}\right\}$$

Now we can define Weber functions

$$f(z) = \zeta_{48}^{-1} \frac{\eta\left(\frac{z+1}{2}\right)}{\eta(z)} = q^{-\frac{1}{48}} \prod_{n\geq 1} (1+q^{n-\frac{1}{2}})$$

$$f_1(z) = \frac{\eta(z/2)}{\eta(z)} = q^{-\frac{1}{48}} \prod (1-q^{n-\frac{1}{2}})$$

$$f_2(z) = \sqrt{2}\, \frac{\eta(2z)}{\eta(z)} = \sqrt{2}\, q^{\frac{1}{24}} \prod_{n\geq 1} (1+q^n)$$

THESE FUNCTIONS SATISFY MANY INTERESTING IDENTITIES

$$f_1(2z) f_2(z) = f(z) f_1(z) f_2(z) = \sqrt{2}$$

The definition of Weber functions may seem quite unnatural at a first glance but these functions arise organically when studying the 2-torsion on the universal elliptic curve over $\mathbb{C}$.

Theorem LET $e_1 = \wp_\tau\left(\frac{\tau}{2}\right)$, $e_2 = \wp_\tau\left(\frac{1}{2}\right)$, $e_3 = \wp_\tau\left(\frac{1+\tau}{2}\right)$, THEN

$$\begin{cases} e_2 - e_1 = \pi^2 \eta(\tau)^4 f(\tau)^8 \\ e_2 - e_3 = \pi^2 \eta(\tau)^4 f_1(\tau)^8 \\ e_3 - e_1 = \pi^2 \eta(\tau)^4 f_2(\tau)^8 \end{cases}$$

Corollary $\gamma_2(\tau) = \dfrac{f(\tau)^{24} - 16}{f(\tau)^8} = \dfrac{f_1(\tau)^{24} + 16}{f_1(\tau)^8} = \dfrac{f_2(\tau)^{24} + 16}{f_1(\tau)^8}$

Corollary

$$f(z+1) = \zeta_{48}^{-1} f(z) \qquad f(-1/z) = f(z)$$

$$f_1(z+1) = \zeta_{48}^{-1} f_1(z) \qquad f_1(-1/z) = f_2(z)$$

$$f_2(z+1) = \zeta_{24} f_2(z) \qquad f_2(-1/z) = f_1(z)$$

It follows that f^6 is a modular function for the congruence sub.

$$\Gamma_0^0(8) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid b \equiv c \equiv 0 \bmod 8 \right\}$$

whereas f_2 is a modular function for $\Gamma(24)$

Geometric Interpretation

We saw before that there exists an isomorphism

$$\begin{aligned} \mathbb{C}/\Lambda_\tau &\longrightarrow E_\tau \\ z &\longmapsto (\wp : \wp' : 1) \end{aligned}$$

This map describes how the algebraic description, as a curve with Weierstrass equation connects to the complex analytic description
$$E_\tau = \mathbb{C}/\Lambda_\tau$$
THIS ALSO SHOWS THAT THE STANDARD BASIS $H^1_{dR}(E_\tau/\mathbb{C}) = \langle \frac{dx}{y}, x\frac{dx}{y} \rangle$ ADMITS A SIMPLE DESCRIPTION ON THE ANALYTIC SIDE AS dz, $\wp dz$.

Given an element $w \in \Lambda_\tau$ we obtain a corresponding element in the singular homology by considering the image of a path from 0 to w in $\mathbb{C}$
$$\begin{aligned} \Lambda_\tau &\longrightarrow H_1(E|\mathbb{C}) \\ w &\longmapsto [\gamma_0^w] \end{aligned}$$
The Poincaré duality pairing may now be computed by calculating the line integrals
$$\int_0^1 dz = 1 \quad \int_0^\tau dz = \tau \quad \text{and} \quad \int_0^1 \wp(z)\,dz \quad \int_0^\tau \wp(z)\,dz$$
The Weierstrass ζ-function provides us with a primitive for $\wp(z)dz$ as $\zeta'(z) = -\wp(z)$, so we see that the periods are described precisely by the failure of ζ to be elliptic and are encoded in the quasi-period map.

Finally, the Weierstrass σ-function is related to differentials of the III kind.

COMPLEX MULTIPLICATION

WE RECALL THAT ORDERS IN QUADRATIC FIELDS GIVE RISE TO CLASS OF LATTICES
$$\mathfrak{a} = \text{proper fractional } \mathcal{O}\text{-ideal.}$$
$$\mathfrak{a} = [\alpha, \beta] \quad \text{SOME } \alpha, \beta \in K$$
NOW WE REGARD K AS A SUBSET OF $\mathbb{C}$: $K \hookrightarrow \mathbb{C} \Rightarrow [\alpha, \beta]$ GIVES RISE TO $\Lambda = \alpha\mathbb{Z} + \beta\mathbb{Z}$ SINCE α AND β ARE INDEPENDENT OVER $\mathbb{R}$.
THEN WE DEFINE THE j-INVARIANT
$$j(\mathfrak{a}) = j(\Lambda_\mathfrak{a})$$

Recall $\wp(z+w) = -\wp(z) - \wp(w) + \frac{1}{4}\left(\frac{\wp'(z) - \wp'(w)}{\wp(z) - \wp(w)}\right)^2$

IF $z = w$ $\quad \wp(2z) = -2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2$

Recall $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$
$$\wp''(z) = 6\wp(z)^2 - \frac{g_2}{2}$$

THUS $\wp(2z) = -2\wp(z) + \dfrac{(12\wp(z)^2 - g_2)^2}{16(4\wp(z)^3 - g_2\wp(z) - g_3)}$

BY INDUCTION $\wp(nz)$ IS A RATIONAL FUNCTION IN $\wp(z)$ FOR ALL n

Theorem LET Λ BE A LATTICE. THEN FOR ALL $\alpha \in \mathbb{C} \setminus \mathbb{Z}$ THE FOLLOWING STATEMENTS ARE EQUIVALENT

I. $\wp(\alpha z)$ IS A RATIONAL FUNCTION IN $\wp(z)$
II. $\alpha\Lambda \subset \Lambda$
III. $\exists\ \mathcal{O} \subseteq K$ IMAGINARY QUADRATIC FIELD SUCH THAT $\alpha \in \mathcal{O}$ AND Λ IS HOMOTHETIC TO A PROPER FRACTIONAL $\mathcal{O}$-IDEAL.

FURTHER, IF THESE CONDITION ARE SATISFIED THEN

$$\wp(\alpha z) = \frac{A(\wp(z))}{B(\wp(z))} \quad \text{WITH } A(x) \text{ AND } B(x)$$

RELATIVELY PRIME POLYNOMIALS SUCH THAT
$$\deg A = \deg B + 1 = [\Lambda : \alpha\Lambda] = N(\alpha)$$

This theorem shows that if an elliptic function has multiplication by some α $\alpha \in \mathbb{C}$, then it has multiplication by an entire order $\mathcal{O} \subset K$.

Remark WE CAN RELATE HOMOTHETY CLASSES OF LATTICES WITH IDEAL CLASS GROUPS
fix $\mathcal{O}$ and choose $\Lambda \subseteq \mathbb{C}$ such that $\mathcal{O}$ = full ring of complex multiplication of Λ
Λ IS A PROPER FRACTIONAL $\mathcal{O}$-IDEAL AND CONVERSELY, ANY PROPER FRACTIONAL $\mathcal{O}$-IDEAL IS A LATTICE WITH CM BY $\mathcal{O}$

TWO PROPER FRACTIONAL $\mathcal{O}$-IDEALS ARE HOMOTHETIC AS LATTICES IF AND ONLY IF THEY ARE IN THE SAME EQUIVALENCE CLASS IN $\mathcal{Cl}(\mathcal{O})$

Corollary THERE IS A BIJECTION

$$\mathcal{Cl}(\mathcal{O}) \longleftrightarrow \left\{ \begin{array}{l} \text{homothety classes} \\ \text{of lattices with CM by } \mathcal{O} \end{array} \right\}$$

example SUPPOSE Λ HAS COMPLEX MULTIPLICATION BY $\sqrt{-3}$
$\Rightarrow \mathcal{O} \subset K$ CONTAINS $\sqrt{-3}$ $\Rightarrow \mathcal{O}$ IS EITHER $\mathbb{Z}[\sqrt{-3}]$ OR $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$
AND THEY BOTH HAVE CLASS NUMBER 1.

THEN THE ONLY LATTICES (UP TO HOMOTHETY) WITH COMPLEX MULTIPLICATION BY $\sqrt{-3}$ ARE $[1, \sqrt{-3}]$ AND $[1, \frac{1+\sqrt{-3}}{2}]$

Question How does complex multiplication affect the j-invariant.

$j(i) = j([1, i])$ $\quad \Lambda = [1, i]$

↳ only lattice, up to homothety with complex multiplication

$$g_3(i\Lambda) = g_3(\Lambda) = i^6 g_3(\Lambda) = -g_3(\Lambda) \Rightarrow g_3(\Lambda) = 0$$

THEN $j(i) = 1728$

SUPPOSE NOW $\omega = \frac{1+\sqrt{-3}}{2}$, THEN $\quad \overline{\omega} = \omega^2 = -1-\omega$

$$\overline{G_{2k}(\Lambda)} = \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{1}{(n + m\overline{\omega})^{2k}} = \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{1}{(n - (1+\omega)m)^{2k}}$$

$$= \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{1}{((n-m) - \omega m)^{2k}} = G_{2k}(\Lambda)$$

↖ m, n runs over all $\mathbb{Z}$

THUS $G_{2k}(\Lambda) \in \mathbb{R} \Rightarrow G_4 \in \mathbb{R}$. NOW

$$\omega G_4 = \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{\omega}{(n + m\omega)^4} = \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \left(\frac{\omega}{n + \omega m}\right)^4 = \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \left(\frac{1}{\frac{n}{\omega} + m}\right)^4$$

$$= \sum_{\substack{n,m \in \mathbb{Z} \\ (n,m) \neq (0,0)}} \frac{1}{(\overline{\omega} n + m)^4} = \overline{G_4} = G_4 \Rightarrow (\omega - 1) G_4 = 0 \Rightarrow G_4 = 0$$

$$\Rightarrow g_4 = 0$$

THUS $j(\omega) = 0$

Theorem $\mathcal{O}$ ORDER IN QUADRATIC IMAGINARY FIELD
$\mathfrak{a}$ = PROPER FRACTIONAL $\mathcal{O}$-IDEAL $\Rightarrow j(\mathfrak{a})$ IS AN ALGEBRAIC NUMBER OF DEGREE AT MOST $h(\mathcal{O})$

Proof $\wp(z; g_2, g_3) = \frac{1}{z^2} + \sum_{n \geq 1} a_n(g_2, g_3) z^{2n}$

By assumption $\forall \alpha \in \mathcal{O}$, $\wp(\alpha z, g_2, g_3) = \frac{A(\wp(z, g_2, g_3))}{B(\wp(z, g_2, g_3))} \in \mathbb{C}((z))$

IN $\mathbb{C}((z))$ elements are $\sum_{n=-M}^{+\infty} b_n z^n$

ON THE OTHER HAND $\wp(\alpha z, g_2, g_3) = \frac{1}{\alpha^2 z^2} + \sum_{n\geq 1} a_n(g_2, g_3)\alpha^{2n} z^{2n}$

NOW FOR $\sigma \in \mathrm{Aut}(\mathbb{C})$

$$(*) \quad \wp(\sigma(\alpha)z, \sigma(g_2), \sigma(g_3)) = \frac{A^\sigma(\wp(z), \sigma(g_2), \sigma(g_3)))}{B^\sigma(\wp(z), \sigma(g_2), \sigma(g_3)))}$$

SINCE $g_2^3 - 27g_3^2 \neq 0 \Rightarrow \sigma(g_2)^3 - 27\sigma(g_3^2) \neq 0$

We'll prove in the next talk that this condition guarantees the existence of a lattice Λ' such that

$$g_2(\Lambda') = g_2' = \sigma(g_2)$$
$$g_3(\Lambda') = g_3' = \sigma(g_3)$$

NOW $(*)$ IMPLIES THAT Λ' HAS CM BY $\sigma(\alpha)$. THEN

$$\mathcal{O} = \sigma(\mathcal{O}) \subset \mathcal{O}'$$

REPLACING σ WITH σ^{-1} $\mathcal{O} \subset \mathcal{O}' \subset \mathcal{O} \Rightarrow \mathcal{O} = \mathcal{O}'$. LOOKING AT EQUATIONS FOR g_2, g_3, WE CONCLUDE

$$j(\Lambda') = \sigma(j(\Lambda))$$

SINCE Λ' HAS CM BY $\mathcal{O}$ WE KNOW THAT THERE ARE ONLY $h(\mathcal{O})$ POSSIBILITIES FOR $j(\Lambda') \Rightarrow$ ONLY $h(\mathcal{O})$ POSSIBILITIES FOR $\sigma(j(\Lambda))$

Since this is true for all $\sigma \in \mathrm{Aut}(\mathbb{C}) \Rightarrow j$ IS AN ALGEBRAIC NUMBER WITH DEGREE $\leq h(\mathcal{O})$.

<u>exercise</u> I WE KNOW $j(\Lambda_1) = j(\Lambda_2) \Leftrightarrow \exists\, \alpha \in \mathbb{C}^\times : \alpha\Lambda_1 = \Lambda_2$
SHOW THAT FOR AN ELLIPTIC CURVE E WITH LATTICE Λ, E CAN BE DEFINED OVER $\mathbb{R} \Leftrightarrow \exists\, \alpha \in \mathbb{C}^\times : \alpha\Lambda = \overline{\alpha\Lambda}$

<u>exercise</u> II LET E BE AN ELLIPTIC CURVE WITH CORRESPONDING LATTICE $\Lambda = \overline{\Lambda}$
SHOW THAT THERE IS A BASIS SUCH THAT $\Lambda = \omega\mathbb{Z} + \tau\mathbb{Z}$
WITH $\omega \in \mathbb{R}$ $\mathrm{Re}(\tau) = \frac{\omega}{2}$ OR $R(\tau) = 0$

BESIDES, SHOW THAT $\mathrm{Re}(\tau) = 0 \Leftrightarrow$ THE CUBIC POLYNOMIALS HAS THREE REAL ROOTS (THUS $E[2] \subset \mathbb{R}$)

Proof $\Delta(\tau) = (2\pi i)^{12} q \prod_{n\geq 1} (1-q^n)^{24}$

$\rightarrow$ FIRST WE USE q-EXPANSION OF THE σ-FUNCTION (Theorem 18.4 Lang's Book "Elliptic Function".

$$\sigma(z,\tau) = (2\pi i)^{-1} e^{\frac{1}{2}\eta(1)z^2} e^{-\pi i z}\left(1-e^{2\pi i z}\right) \prod_{n\geq 1} \frac{(1-q_\tau^n q_z)(1-q_\tau^n/q_z)}{(1-q_\tau^n)^2}$$

$\rightarrow$ RECALL THAT

- $\sigma(z+w) = \sigma(z)\psi(w)\exp\left(\eta(w)\left(z+\frac{w}{2}\right)\right)$
- $\wp(z_1)-\wp(z_2) = \dfrac{\sigma(z_1+z_2)\sigma(z_1-z_2)}{\sigma(z_1)^2\sigma(z_2)^2}$

$q_\tau = e^{2\pi i \tau}$

$q_z = e^{2\pi i z}$

$$e_2 - e_1 = \wp\left(\tfrac{1}{2}\right) - \wp\left(\tfrac{\tau}{2}\right) = \frac{\sigma\left(\frac{\tau+1}{2}\right)\sigma\left(\frac{1-\tau}{2}\right)}{\sigma\left(\frac{1}{2}\right)^2\sigma\left(\frac{\tau}{2}\right)^2} \underset{\uparrow}{=} e^{-\eta(1)\frac{\tau}{2}} \frac{\sigma\left(\frac{\tau+1}{2}\right)^2}{\sigma\left(\frac{1}{2}\right)^2\sigma\left(\frac{\tau}{2}\right)^2}$$

σ is an odd function

$$\sigma\left(\tfrac{1-\tau}{2}\right) = -\sigma\left(\tfrac{\tau-1}{2}\right) = -\sigma\left(\tfrac{\tau+1}{2} \underset{-1\in\Lambda\setminus 2\Lambda}{-1}\right) = \sigma\left(\tfrac{\tau+1}{2}\right) e^{-\eta(1)\frac{\tau}{2}}$$

$$e_2 - e_3 = \wp\left(\tfrac{1}{2}\right) - \wp\left(\tfrac{\tau+1}{2}\right) = \frac{\sigma\left(\frac{\tau}{2}+1\right)\sigma\left(-\frac{\tau}{2}\right)}{\sigma\left(\frac{1}{2}\right)^2\sigma\left(\frac{\tau+1}{2}\right)^2} \underset{\uparrow}{=} e^{\eta(1)(\tau+1)\frac{1}{2}} \frac{\sigma\left(\frac{\tau}{2}\right)^2}{\sigma\left(\frac{1}{2}\right)^2\sigma\left(\frac{\tau+1}{2}\right)^2}$$

$$\sigma\left(\tfrac{\tau}{2}\underset{1\in\Lambda\setminus 2\Lambda}{+1}\right) = -e^{\eta(1)\frac{1}{2}(\tau+1)}\sigma\left(\tfrac{\tau}{2}\right)$$

$$e_3 - e_1 = \wp\left(\tfrac{\tau+1}{2}\right) - \wp\left(\tfrac{\tau}{2}\right) = \frac{\sigma\left(\tau+\frac{1}{2}\right)\sigma\left(\frac{1}{2}\right)}{\sigma\left(\frac{\tau+1}{2}\right)^2\sigma\left(\frac{\tau}{2}\right)^2} = -e^{\eta(\tau)\frac{\tau+1}{2}} \frac{\sigma\left(\frac{1}{2}\right)^2}{\sigma\left(\frac{\tau+1}{2}\right)^2\sigma\left(\frac{\tau}{2}\right)^2}$$

$$\Delta(q) = 16(e_1-e_2)^2(e_1-e_3)^2(e_2-e_3)^2 =$$

$$= 16\, e^{-\eta(1)\tau+\eta(1)(\tau+1)+\eta(\tau)(\tau+1)} \left(\frac{\cancel{\sigma\left(\frac{1}{2}\right)^2}\,\cancel{\sigma\left(\frac{\tau}{2}\right)^2}\,\cancel{\sigma\left(\frac{\tau+1}{2}\right)^2}}{\cancel{\sigma\left(\frac{1}{2}\right)^2}\,\cancel{\sigma\left(\frac{\tau}{2}\right)^2}\,\sigma\left(\frac{1}{2}\right)^2\,\cancel{\sigma\left(\frac{\tau+1}{2}\right)^2}\,\sigma\left(\frac{\tau+1}{2}\right)^2\,\sigma\left(\frac{\tau}{2}\right)^2}\right)^2$$

$$\underset{\uparrow}{=} 16\, e^{-\cancel{\eta(1)\tau}+\cancel{\eta(1)\tau}+\eta(1)+\tau^2\eta(1)-2\pi i\tau+\tau\eta(1)-2\pi i} \frac{1}{\sigma\left(\frac{1}{2}\right)^4\sigma\left(\frac{\tau+1}{2}\right)^4\sigma\left(\frac{\tau}{2}\right)^4}$$

Recall $\tau\eta(1)-\eta(\tau) = 2\pi i \Rightarrow \eta(\tau) = \tau\eta(1) - 2\pi i$

$$= 16\, \frac{e^{\eta(1)[\tau^2+\tau+1]-2\pi i\tau-2\pi i}}{\sigma\left(\frac{1}{2}\right)^4\sigma\left(\frac{\tau+1}{2}\right)^4\sigma\left(\frac{\tau}{2}\right)^4}$$

NOW WE USE THE q-EXPANSION OF THE σ FUNCTION. THE DENOMINATOR BECOMES

$$\sigma\left(\tfrac{1}{2}\right)^4\sigma\left(\tfrac{\tau+1}{2}\right)^4\sigma\left(\tfrac{\tau}{2}\right)^4 = \Bigg[(2\pi i)^{-3} e^{\frac{1}{2}\eta(1)\left[\frac{1}{4}+\frac{\tau^2+2\tau+1}{4}+\frac{\tau^2}{4}\right]} e^{-\pi i\left(\frac{1}{2}+\frac{\tau+1}{2}+\frac{\tau}{2}\right)}$$

$$\left(1-e^{\pi i}\right)\left(1-e^{\pi i\tau}\right)\left(1-e^{\pi i(\tau+1)}\right) \prod_{n\geq 1} \frac{(1-q_\tau^n q_{\frac{1}{2}})(1-q_\tau^n/q_{\frac{1}{2}})(1-q_\tau^n q_{\frac{\tau+1}{2}})(1-q_\tau^n/q_{\frac{\tau+1}{2}})(1-q_\tau^n q_{\tau/2})(1-q_\tau^n/q_{\tau/2})}{(1-q_\tau^n)^6}\Bigg]^4 =$$

OBSERVE THAT $q_{\frac{1}{2}} = e^{2\pi i \frac{1}{2}} = e^{\pi i} = -1$, $q_{\tau/2} = e^{\pi i \tau} = q_\tau^{\frac{1}{2}}$, $q_{\frac{\tau+1}{2}} = -q_\tau^{\frac{1}{2}}$, THUS

$$\mathrm{den} = \Bigg[(2\pi i)^{-3} e^{\frac{1}{4}\eta(1)[\tau^2+\tau+1] - \pi i - \pi i \tau} \cdot 2 \cdot (1-q_\tau^{1/2})(1+q_\tau^{1/2}) \cdot \prod_{n\geq 1} \frac{(1+q_\tau^n)(1+q_\tau^n)(1-q_\tau^{n+\frac{1}{2}})(1-q_\tau^{n-\frac{1}{2}})(1+q_\tau^{n+\frac{1}{2}})(1+q_\tau^{n-\frac{1}{2}})}{(1-q_\tau^n)^6} \Bigg]^4$$

$$\Delta(q) = \cancel{16}(2\pi i)^{12} \frac{e^{\cancel{\eta(1)[\tau^2+\tau+1]}-2\pi i - 2\pi i \tau} \prod_{n\geq 1}(1-q_\tau^n)^{24}}{\cancel{16} e^{\cancel{\eta(1)[\tau^2+\tau+1]}-4\pi i - 4\pi i \tau} (1-q_\tau)^4 \prod_{n\geq 1} (1+q_\tau^n)^8 (1-q_\tau^{2n+1})^4 (1-q_\tau^{2n-1})^4}$$

$$= (2\pi i)^{12} \underbrace{e^{2\pi i \tau}}_{q_\tau} \frac{\prod_{n\geq 1}(1-q_\tau^n)^{24}}{(1-q_\tau)^4 \prod_{n\geq 1} (1+q_\tau^n)^4 (1-q_\tau^{2n+1})^4 (1+q_\tau^n)^4 (1-q_\tau^{2n-1})^4}$$

NOW

$$\prod_{n\geq 1}(1-q_\tau^n)^2 \cdot \prod_{n\geq 1}(1+q_\tau^n)^2 (1-q_\tau^{2n+1})(1-q_\tau^{2n-1}) =$$

$$= \prod_{n\geq 1}(1-q_\tau^{2n})^2 (1-q_\tau^{2n+1})(1-q_\tau^{2n-1}) = (1-q_\tau)(1-q_\tau^2)^2(1-q_\tau^3)(1-q_\tau^3)$$

$$(1-q_\tau^4)^2(1-q_\tau^5)(1-q_\tau^5)(1-q_\tau^6)(1-q_\tau^7)\ldots$$

$$= \frac{\prod_{n\geq 1}(1-q_\tau^n)^2}{(1-q_\tau)}$$ since odd exponents appear twice (except 1) and so do even exponents

$$\Rightarrow \prod (1+q_\tau^n)^2(1-q_\tau^{2n+1})(1-q_\tau^{2n-1}) = \frac{1}{(1-q_\tau)}$$

HENCE

$$\Delta(q) = (2\pi i)^{12} q_\tau \frac{\prod_{n\geq 1}(1-q_\tau)^{24}}{\cancel{(1-q_\tau)^4} \left[\frac{1}{\cancel{(1-q_\tau)}}\right]^4} = (2\pi i)^{12} q \prod_{n\geq 1}(1-q)^{24}$$